Durham University
Pavel Tumarkin Michaelmas 2016

**Differential Geometry III, Solutions 9 (Week 9)**

# First fundamental form

**9.1.** Find the coefficients of the first fundamental forms of:

(a) the *catenoid* parametrized by

$$\boldsymbol{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v), \qquad (u,v) \in U := (0, 2\pi) \times \mathbb{R};$$

(b) the *helicoid* parametrized by

$$\widetilde{\boldsymbol{x}}(u,v) = (-\sinh v \sin u, \sinh v \cos u, -u), \qquad (u,v) \in U;$$

(c) the surface $S_\vartheta$ (for some $\vartheta \in \mathbb{R}$) parametrized by

$$\boldsymbol{y}_\vartheta(u,v) = (\cos\vartheta)\boldsymbol{x}(u,v) + (\sin\vartheta)\widetilde{\boldsymbol{x}}(u,v), \qquad (u,v) \in U.$$

*Solution:*

(a) We have

$$\begin{aligned} \partial_u \boldsymbol{x}(u,v) &= (-\cosh v \sin u, \cosh v \cos u, 0), \\ \partial_v \boldsymbol{x}(u,v) &= (\sinh v \cos u, \sinh v \sin u, 1). \end{aligned}$$

This implies that

$$\begin{aligned} E(u,v) &= (-\cosh)^2 v \sin^2 u + \cosh^2 v \cos^2 u = \cosh^2 v, \\ F(u,v) &= 0, \\ G(u,v) &= \sinh^2 v \cos^2 u + \sinh^2 v \sin^2 u + 1 = \sinh^2 v + 1 = \cosh^2 v, \end{aligned}$$

i.e., the first fundamental form at $\boldsymbol{x}(u,v)$ is just a multiple of the standard inner product in $\mathbb{R}^2$ by the factor $\cosh^2 v$.

(b) We have

$$\begin{aligned} \partial_u \widetilde{\boldsymbol{x}}(u,v) &= (-\sinh v \cos u, -\sinh v \sin u, -1), \\ \partial_v \widetilde{\boldsymbol{x}}(u,v) &= (-\cosh v \sin u, \cosh v \cos u, 0). \end{aligned}$$

This implies that

$$\begin{aligned} \widetilde{E}(u,v) &= (-\sinh)^2 v \cos^2 u + (-\sinh)^2 v \sin^2 u + (-1)^2 = \cosh^2 v, \\ \widetilde{F}(u,v) &= 0, \\ \widetilde{G}(u,v) &= (-\cosh)^2 v \sin^2 u + \cosh^2 v \cos^2 u = \cosh^2 v, \end{aligned}$$

i.e., the first fundamental form at $\widetilde{\boldsymbol{x}}(u,v)$ is again just a multiple of the standard inner product in $\mathbb{R}^2$ by the factor $\cosh^2 v$.

(c) Now we choose

$$\boldsymbol{y}_\vartheta(u,v) = \cos\vartheta \boldsymbol{x}(u,v) + \sin\vartheta \widetilde{\boldsymbol{x}}(u,v).$$

We obviously have

$$\begin{aligned}\partial_u \boldsymbol{y}_\vartheta &= \cos\vartheta \partial_u \boldsymbol{x} + \sin\vartheta \partial_u \widetilde{\boldsymbol{x}},\\ \partial_v \boldsymbol{y}_\vartheta &= \cos\vartheta \partial_v \boldsymbol{x} + \sin\vartheta \partial_v \widetilde{\boldsymbol{x}}.\end{aligned}$$

We easily check that $\langle \partial_u \boldsymbol{x}, \partial_u \widetilde{\boldsymbol{x}}\rangle = 0 = \langle \partial_v \boldsymbol{x}, \widetilde{\partial}_v \boldsymbol{x}\rangle$ and

$$\langle \partial_u \boldsymbol{x}, \widetilde{\partial}_v \boldsymbol{x}\rangle + \langle \partial_v \boldsymbol{x}, \widetilde{\partial}_u \boldsymbol{x}\rangle = \cosh^2 v - (\sinh^2 v + 1) = 0.$$

This implies that

$$\begin{aligned}\langle \partial_u \boldsymbol{y}_\vartheta, \partial_u \boldsymbol{y}_\vartheta\rangle &= \cos^2\vartheta E + \sin^2\vartheta \widetilde{E} + 2\sin\vartheta\cos\vartheta \langle \partial_u \boldsymbol{x}, \widetilde{\partial}_u \boldsymbol{x}) = \cosh^2 v,\\ \langle \partial_u \boldsymbol{y}_\vartheta, \partial_v \boldsymbol{y}_\vartheta\rangle &= \cos^2\vartheta F + \sin^2\vartheta \widetilde{F} + \sin\vartheta\cos\vartheta (\langle \partial_u \boldsymbol{x}, \widetilde{\partial}_v \boldsymbol{x}\rangle + \langle \partial_v \boldsymbol{x}, \widetilde{\partial}_u \boldsymbol{x}\rangle)\\ &= \cos^2\vartheta \cdot 0 + \sin^2\vartheta \cdot 0 + \sin\vartheta\cos\vartheta \cdot 0 = 0,\\ \langle \partial_v \boldsymbol{y}_\vartheta, \partial_v \boldsymbol{y}_\vartheta\rangle &= \cos^2\vartheta G + \sin^2\vartheta \widetilde{G} + 2\sin\vartheta\cos\vartheta \langle \partial_v \boldsymbol{x}, \partial_v \widetilde{\boldsymbol{x}}) = \cosh^2 v,\end{aligned}$$

i.e., the first fundamental form at $\boldsymbol{y}_\vartheta(u,v)$ is again just a multiple of the standard inner product in $\mathbb{R}^2$ by the factor $\cosh^2 v$.

**9.2.** Find the coefficients of the first fundamental form of

(a) $S^2(1)$ with respect to the local parametrization $\boldsymbol{x}$ defined in Exercise 6.2;

(b) the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x\sin z - y\cos z = 0\}$$

parametrized by

$$\boldsymbol{x}(u,v) = (\sinh v \cos u, \sinh v \sin u, u)$$

*Solution:*

(a) We have

$$\boldsymbol{x}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

Therefore,

$$\begin{aligned}\partial_u \boldsymbol{x}(u,v) &= \left(\frac{2(1-u^2+v^2)}{(u^2+v^2+1)^2}, \frac{-4uv}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2}\right),\\ \partial_v \boldsymbol{x}(u,v) &= \left(\frac{-4uv}{(u^2+v^2+1)^2}, \frac{2(1+u^2-v^2)}{(u^2+v^2+1)^2}, \frac{4v}{(u^2+v^2+1)^2}\right).\end{aligned}$$

Hence,

$$\begin{aligned}E &= \langle \partial_u \boldsymbol{x}\,,\, \partial_u \boldsymbol{x}\rangle\\ &= \frac{4(1-u^2+v^2)^2 + 16u^2(v^2+1)}{(u^2+v^2+1)^4}\\ &= \frac{4}{(u^2+v^2+1)^2},\\ F &= \langle \partial_u \boldsymbol{x}\,,\, \partial_v \boldsymbol{x}\rangle\\ &= \frac{-8uv(1-u^2+v^2) - 8uv(1+u^2-v^2) + 18uv}{(u^2+v^2+1)^4}\\ &= 0,\\ G &= \langle \partial_v \boldsymbol{x}\,,\, \partial_v \boldsymbol{x}\rangle\\ &= \frac{4(1+u^2-v^2)^2 + 16v^2(u^2+1)}{(u^2+v^2+1)^4}\\ &= \frac{4}{(u^2+v^2+1)^2}.\end{aligned}$$

(b) We have

$$\begin{aligned}\partial_u \boldsymbol{x}(u,v) &= (-\sinh v \sin u, \sinh v \cos u, 1),\\ \partial_v \boldsymbol{x}(u,v) &= (\cosh v \cos u, \cosh v \sin u, 0).\end{aligned}$$

This implies that

$$\begin{aligned}E(u,v) &= (-\sinh)^2 v \sin^2 u + \sinh^2 v \cos^2 u + 1^2 = \cosh^2 v,\\ F(u,v) &= 0,\\ G(u,v) &= \cosh^2 v \cos^2 u + \cosh^2 v \sin^2 u = \cosh^2 v.\end{aligned}$$

**9.3.** Let $U = \mathbb{R}\times(0,\infty)$, and let $\boldsymbol{x} : U \to \mathbb{R}^n$ be a parametrization of a surface $\mathbb{H}$ in $\mathbb{R}^2$ with corresponding coefficients of the first fundamental form $E(u,v) = G(u,v) = 1/v^2$ and $F(u,v) = 0$ for all $(u,v) \in U$. Then $\mathbb{H}$ is called the *hyperbolic plane*. For $r > 0$ denote by $\boldsymbol{\alpha} : (0,\pi) \to \mathbb{H}$ the curve given by

$$\boldsymbol{\alpha}(t) = \boldsymbol{x}(r\cos t, r\sin t).$$

Show that the length of $\boldsymbol{\alpha}$ in $\mathbb{H}$ from $\boldsymbol{\alpha}(\pi/6)$ to $\boldsymbol{\alpha}(5\pi/6)$ is equal to

$$\int_{\pi/6}^{5\pi/6} \frac{1}{\sin t}\,\mathrm{d}t.$$

(In fact, $\boldsymbol{\alpha}$ is the curve of shortest length between its endpoints.) Now take $r = \sqrt{2}$ and find the angle of intersection of $\boldsymbol{\alpha}$ with the curve $\boldsymbol{\beta}(s) = \boldsymbol{x}(1,s)$ at their point of intersection.

*Solution:*

Let $\boldsymbol{\alpha}(t) = \boldsymbol{x}(c\cos t, c\sin t)$, $\pi/6 \le t \le 5\pi/6$ be a curve in the hyperbolic plane. Let $u(t) = c\cos t$ and $v(t) = c\sin t$. The lenght of $\boldsymbol{\alpha}$ is

$$\begin{aligned}l(\boldsymbol{\alpha}) &= \int_{\pi/6}^{5\pi/6} \|\boldsymbol{\alpha}'(t)\|\,\mathrm{d}t\\ &= \int_{\pi/6}^{5\pi/6} (Eu'^2 + 2Fu'v' + Gv'^2)^{1/2}\,\mathrm{d}t\\ &= \int_{\pi/6}^{5\pi/6} \left(\frac{1}{c^2\sin^2 t}\left(c^2\sin^2 t + c^2\cos^2 t\right)\right)^{1/2}\,\mathrm{d}t\\ &= \int_{\pi/6}^{5\pi/6} \frac{1}{\sin t}\,\mathrm{d}t.\end{aligned}$$

If $c = \sqrt{2}$ then $\boldsymbol{\alpha}(t)$ intersects $\boldsymbol{\beta}(s) = \boldsymbol{x}(1,s)$ at points where $u(t) = 1$ and $v(t) = s$. Solving theses equations gives $\cos t = \sin t = \sqrt{2}/2$, so $t = t_0 = \pi/4$ and $s = s_0 = 1$.

At the point of intersection $\partial_u \boldsymbol{x}(1,1)$, we have $E = G = 1$ and $F = 0$.

At $t_0 = \pi/4$,

$$\begin{aligned}\boldsymbol{\alpha}'(t_0) &= \partial_u \boldsymbol{x}(1,1)u'(t_0) + \partial_v \boldsymbol{x}(1,1)v'(t_0)\\ &= -\sqrt{2}\partial_u \boldsymbol{x}(1,1)\sin t_0 + \sqrt{2}\partial_v \boldsymbol{x}(1,1)\cos t_0\\ &= -\partial_u \boldsymbol{x}(1,1) + \partial_v \boldsymbol{x}(1,1).\end{aligned}$$

Similarly, at $s_0 = 1$, $\boldsymbol{\beta}'(s_0) = \partial_v \boldsymbol{x}(1,1)$. Therefore, the angle of intersection of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ at their point of intersection is

$$\cos\vartheta = \frac{\langle \boldsymbol{\alpha}'(t_0), \boldsymbol{\beta}'(s_0)\rangle}{\|\boldsymbol{\alpha}'(t_0)\|\|\boldsymbol{\beta}'(s_0)\|} = \frac{\langle -\partial_u \boldsymbol{x}(1,1) + \partial_v \boldsymbol{x}(1,1), \partial_v \boldsymbol{x}(1,1)\rangle}{\sqrt{1+1}\sqrt{1}} = \frac{1}{\sqrt{2}}.$$

Thus, $\vartheta = \pi/4$.

**9.4.** Let $S$ be a surface parametrized by

$$\boldsymbol{x}(u,v) = (u\cos v, u\sin v, \log\cos v + u), \qquad (u,v) \in U := \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

For $c \in (-\pi/2, \pi/2)$, let $\boldsymbol{\alpha}_c$ be the curve given by $\boldsymbol{\alpha}_c u = \boldsymbol{x}(u,c)$. Show that the length of $\boldsymbol{\alpha}_c$ from $u = u_0$ to $u = u_1$ does not depend on $c$.

*Solution:* The length of $\boldsymbol{\alpha}_c$ is given by

$$l(\boldsymbol{\alpha}_c) = \int_{u_0}^{u_1} \|\boldsymbol{\alpha}_c'(u)\| \, \mathrm{d}u = \int_{u_0}^{u_1} \|\partial_u \boldsymbol{x}(u,c)\| \, \mathrm{d}u = \int_{u_0}^{u_1} \sqrt{E(u,c)} \, \mathrm{d}u$$

We have

$$\partial_u \boldsymbol{x}(u,v) = (\cos v, \sin v, 1),$$

so

$$E = \langle \partial_u \boldsymbol{x}, \partial_u \boldsymbol{x} \rangle = \cos^2 v + \sin^2 v + 1 = 2$$

Thus,

$$l(\boldsymbol{\alpha}_c) = \int_{u_0}^{u_1} \sqrt{2} \, \mathrm{d}u = \sqrt{2}(u_1 - u_0)$$