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Differential Geometry III, Term 1 (Section 2)

2 Regular curves in $\mathbb{R}^n$

Definition 2.1.

(a) A *smooth curve* in $\mathbb{R}^n$ is a smooth (that is, infinitely differentiable) map

$$\boldsymbol{\alpha}\colon I \to \mathbb{R}^n,$$

where I is an open interval of $\mathbb{R}$ (so I could be (a,b) or $(-\infty,b)$ or $(a,+\infty)$ or $\mathbb{R}$).

(b) The *image*, $\boldsymbol{\alpha}(I)$, of I under $\boldsymbol{\alpha}$ is called the *trace* of $\boldsymbol{\alpha}$. The variable $u \in I$ is called the *parameter* of $\boldsymbol{\alpha}$.

(c) If we write

$$\boldsymbol{\alpha}(u) = (\alpha_1(u), \alpha_2(u), \ldots, \alpha_n(u))$$

then each $\alpha_i\colon I \to \mathbb{R}$ is smooth. The vector

$$\boldsymbol{\alpha}'(u) = (\alpha_1'(u), \alpha_2'(u), \ldots, \alpha_n'(u))$$

is the *tangent vector* to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u)$.

(d) The curve $\boldsymbol{\alpha}$ is *regular* if $\boldsymbol{\alpha}'(u) \neq \mathbf{0} = (0,\ldots,0)$ for all $u \in I$. The curve $\boldsymbol{\alpha}$ is *singular* at $\boldsymbol{\alpha}(u)$ if $\boldsymbol{\alpha}'(u) = \mathbf{0}$.

(e) If $\boldsymbol{\alpha}$ is a regular curve, we define the unit tangent vector

$$\boldsymbol{t}(u) = \frac{\boldsymbol{\alpha}'(u)}{\|\boldsymbol{\alpha}'(u)\|}.$$

If we want to stress that $\boldsymbol{t}$ is the unit tangent vector of the curve $\boldsymbol{\alpha}$, we also write $\boldsymbol{t}_{\boldsymbol{\alpha}}$.

(f) If $\|\boldsymbol{\alpha}'(u)\| = 1$ for all $u \in I$ then $\boldsymbol{\alpha}$ is called *unit speed.*

Example 2.2.

(a) *The unit circle.* $\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (\cos u, \sin u)$. $\boldsymbol{\alpha}$ is smooth and unit speed.

(b) *The helix.* $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^3$, $\boldsymbol{\alpha}(u) = (\cos u, \sin u, u)$. $\boldsymbol{\alpha}$ is smooth and regular.

(c) *The cusp.* $\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha} = (u^3, u^2)$ so $\boldsymbol{\alpha}$ is smooth. But $\boldsymbol{\alpha}'(s) = (3u^2, 2u)$, so $\boldsymbol{\alpha}'(0) = (0,0)$.

(d) *The node.* $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (u^3 - u, u^2 - 1)$. $\boldsymbol{\alpha}$ is smooth and regular but not injective, since $\boldsymbol{\alpha}(-1) = \boldsymbol{\alpha}(1)$.

Definition 2.3. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$ be a smooth and regular curve. A *change of parameter* for $\boldsymbol{\alpha}$ is a function $h\colon J \longrightarrow I$ where J is an open interval of $\mathbb{R}$ satisfying

(a) h is smooth;

(b) $h'(t) \neq 0$ for all $t \in J$;

(c) $h(J) = I$.

Remark. $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \circ h \colon J \longrightarrow \mathbb{R}^n$ is a smooth curve with the same trace as $\boldsymbol{\alpha}$.

Example 2.4. In the Example 2.2(a) take $J = \mathbb{R}$, $h(v) = 2v$. Then

$$\tilde{\boldsymbol{\alpha}}(v) = (\boldsymbol{\alpha} \circ h)(v) = \boldsymbol{\alpha}(2v) = (\cos 2v, \sin 2v).$$

Definition 2.5. The *arc length* of a curve $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^n$, measured from a point $\boldsymbol{\alpha}(u_0)$ for some $u_0 \in I$, is

$$\ell(u) := \int_{u_0}^{u} \|\boldsymbol{\alpha}'(v)\| \, \mathrm{d}v.$$

Remark. If $\boldsymbol{\alpha}$ is unit speed ($\|\boldsymbol{\alpha}'(u)\| = 1$), then

$$\ell(u) = \int_{u_0}^{u} \|\boldsymbol{\alpha}'(s)\| \, \mathrm{d}s = u - u_0.$$

So the parameter u measures the arc length (up to an additive constant) and is called *arc length parameter*, $\boldsymbol{\alpha}$ is *parametrized by arc length.*

Proposition 2.6. Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^n$ be a smooth and regular curve. Choose $u_o \in I$, and let $\ell \colon I \longrightarrow \mathbb{R}$ be the arc length of $\boldsymbol{\alpha}$ w.r. to u_0. Define $J = \ell(I)$. Then ℓ^{-1} is a parameter change, and

$$\boldsymbol{\beta} = \boldsymbol{\alpha} \circ \ell^{-1} \colon J \longrightarrow \mathbb{R}^n$$

is parametrized by arc length.

Example 2.7. *The catenary.*

$$\boldsymbol{\alpha} \colon \mathbb{R} \longrightarrow \mathbb{R}^2, \quad \boldsymbol{\alpha}(u) = (u, \cosh u) \quad \Rightarrow \boldsymbol{\alpha}'(u) = (1, \sinh u)$$

$\boldsymbol{\alpha}$ is regular, $\|\boldsymbol{\alpha}'(u)\| = \sqrt{1 + \sinh^2 u} = \cosh u$,

$$s = \ell(u) = \int_0^u \|\boldsymbol{\alpha}'(t)\| \, \mathrm{d}t = \int_0^u \cosh t \, \mathrm{d}t = \sinh u$$

where we fixed $u_0 = 0$, and thus $u = \ell^{-1}(s) = \sinh^{-1} s$. So the arc-length parametrization of the catenary is

$$\boldsymbol{\beta} = \boldsymbol{\alpha}(\ell^{-1}(s)) = \big(\ln(s + \sqrt{s^2 + 1}), \cosh(\ln(s + \sqrt{s^2 + 1}))\big).$$