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Michaelmas 2016

Differential Geometry III, Term 1 (Section 5)

5 A bit of Analysis (should have been a reminder)

We consider the Euclidean space

$$\mathbb{R}^n = \left\{ \boldsymbol{x} = (x_1, \ldots, x_n) \,\middle|\, x_i \in \mathbb{R}, i = 1, \ldots, n \right\}$$

Definition 5.1.

(a) A *ball of radius* $r > 0$ *with center* $\boldsymbol{a} \in \mathbb{R}^n$ in $\mathbb{R}^n$ is defined by

$$B_r(\boldsymbol{a}) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \|\boldsymbol{x} - \boldsymbol{a}\| = \sqrt{(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2} < r \right\}.$$

(b) A subset $U \subset \mathbb{R}^n$ is called *open*, if for any $\boldsymbol{y} \in U$ there exists $r > 0$ such that $B_r(\boldsymbol{y}) \subset U$, i.e.

$$\forall\, \boldsymbol{y} \in U \;\exists\, r > 0 : \quad B_r(\boldsymbol{y}) \subset U.$$

Example 5.2.

(a) Interval $(a, b) \subset \mathbb{R}$ is open.

(b) Closed interval $[a, b] \subset \mathbb{R}$ is not open.

(c) The ball $B_r(\boldsymbol{a})$ is an open subset of $\mathbb{R}^n$ for any $\boldsymbol{a} \in \mathbb{R}^n$ and $r > 0$.

(d) The *(open) cube* $(a_1, b_1) \times \ldots \times (a_n, b_n)$ is an open subset for any a_i, $b_i \in \mathbb{R}$ with $a_i < b_i$. Note that for $n = 1$, a cube is an interval, and for $n = 2$, a cube is a rectangle (without the boundary).

(e) The entire space $\mathbb{R}^n$ and the empty set $\emptyset$ are open.

Now let $U \subset \mathbb{R}^n$ be open, $\boldsymbol{f} \colon U \longrightarrow \mathbb{R}^m$ be a map, i.e.,

$$\boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} f_1(u_1, \ldots, u_n) \\ \vdots \\ f_m(u_1, \ldots, u_n) \end{pmatrix}$$

for any $\boldsymbol{u} = (u_1, \ldots, u_n) \in U$. We say that $\boldsymbol{f}$ is *smooth* if the (scalar) functions $f_i \colon U \longrightarrow \mathbb{R}$ are smooth for all $i = 1, \ldots m$, i.e., if all partial derivatives of all order exist and are continuous.

Example 5.3.

(a) $\boldsymbol{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ ($U = \mathbb{R}^2$, $n = 2$, $m = 3$) with

$$\boldsymbol{f}(u_1, u_2) = \begin{pmatrix} u_1 \\ u_2 \\ u_1^2 + u_2^2 \end{pmatrix}$$

is a smooth map.

(b) $\boldsymbol{f}\colon B_1(\mathbf{0}) \longrightarrow \mathbb{R}^3$ ($U = B_1(\mathbf{0}) \subset \mathbb{R}^2$, $n = 2$, $m = 3$) with

$$\boldsymbol{f}(u_1, u_2) = \begin{pmatrix} u_1 \\ u_2 \\ \sqrt{1 - u_1^2 - u_2^2} \end{pmatrix}$$

is a smooth map as well.

For (scalar) functions, even of more than one variable, we know how to derive, e.g., if $f(x, y) = x^2 y + 3y^3$, then

$$\frac{\partial f}{\partial x}(x, y) = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = x^2 + 9y^2.$$

Definition 5.4. Let $U \subset \mathbb{R}^n$ be open, let $\boldsymbol{f}\colon U \longrightarrow \mathbb{R}^m$ be a smooth map and let $\boldsymbol{p} \in U$. The *Jacobi matrix* of $\boldsymbol{f}$ at $\boldsymbol{p}$ is the $(m \times n)$-matrix given by

$$J_{\boldsymbol{p}}\boldsymbol{f} := \begin{pmatrix} \partial_1 f_1(\boldsymbol{p}) & \dots & \partial_n f_1(\boldsymbol{p}) \\ \vdots & & \vdots \\ \partial_1 f_m(\boldsymbol{p}) & \dots & \partial_n f_m(\boldsymbol{p}) \end{pmatrix} \qquad \text{where} \quad \partial_i f_j(\boldsymbol{p}) := \frac{\partial}{\partial u_i} f_j(u)\Big|_{u=p}, \qquad i = 1, \dots, n.$$

The *derivative* of $\boldsymbol{f}$ at p is the linear map

$$\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad h \mapsto (\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(h) = J_{\boldsymbol{p}}\boldsymbol{f} \cdot h$$

Note that the Jacobi matrix is just the matrix representation of the derivative in the standard basis.

Remark. Since $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}$ is linear, its image (range) $(\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\mathbb{R}^n)$ is a vector subspace of $\mathbb{R}^m$, spanned by

$$\{(\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\boldsymbol{e}_1), \dots, (\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\boldsymbol{e}_n)\},$$

where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is the standard basis in $\mathbb{R}^n$. Observe that

$$(\partial_i \boldsymbol{f}(\boldsymbol{p}) :=)(\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\boldsymbol{e}_i) = \begin{pmatrix} \partial_i f_1(\boldsymbol{p}) \\ \vdots \\ \partial_i f_m(\boldsymbol{p}) \end{pmatrix}$$

which is just the i^{th} column of the Jacobi matrix $J_{\boldsymbol{p}}\boldsymbol{f}$.

Example 5.5.

(a) $\boldsymbol{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\boldsymbol{f}(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix} \quad \text{then} \quad J_{(u,v)}\boldsymbol{f} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 2v \end{pmatrix}.$$

At $\boldsymbol{p} = (0, 0)$, the image of $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}$ is spanned by $(1, 0, 0)$ and $(0, 1, 0)$.

(b) $\boldsymbol{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$,

$$\boldsymbol{f}(u,v) = \begin{pmatrix} u \\ v^2 \\ uv \end{pmatrix} \quad \text{then} \quad J_{(u,v)}\boldsymbol{f} = \begin{pmatrix} 1 & 0 \\ 0 & 2v \\ v & u \end{pmatrix}.$$

At $\boldsymbol{p} = (0,0)$, the image of $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}$ is spanned by $\{(1,0,0),(0,0,0)\}$, i.e., by $(1,0,0)$ (the x-axis).

(c) $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$f(x,y,z) := 2x^2 + y^2 - z^2, \qquad J_{(x,y,z)}f = (4x, 2y, -2z)$$

(the *gradient* of f). Note that the Jacobi matrix of a scalar function is just the gradient. Here, the image of $\mathrm{d}_{\boldsymbol{p}}f$ is either $\mathbb{R}$ (if $(x,y,z) \neq \mathbf{0}$) or $\{0\}$ (if $(x,y,z) = \mathbf{0}$).

Let us finally motivate the *implicit function theorem*

Example 5.6. Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by $f(u,v) = u^2 + v^2$. We want to solve the equation

$$f(u,v) = c$$

near some point $(a,b) \in \mathbb{R}^2$ for $c := f(a,b) \geq 0$, i.e., we look for a function $g(u) = v$ such that $f(u, g(u)) = c$. The implicit function tells us that if $\partial_v f(u_0, v_0) \neq 0$ then this is possible. Here, $\partial_v f(a,b) = 2b$, and a simple calculation shows that

$$f(u,v) = c \iff v = \begin{cases} \sqrt{c - u^2}, & \text{if } b > 0, \\ -\sqrt{c-u^2}, & \text{if } b < 0. \end{cases}$$

Theorem 5.7 (Implicit function theorem)**.** Let $W \subset \mathbb{R}^p \times \mathbb{R}^m$ be open and $\boldsymbol{f}\colon W \longrightarrow \mathbb{R}^m$ be smooth. Let $(\boldsymbol{a}, \boldsymbol{b}) \in W$ ($\boldsymbol{a} \in \mathbb{R}^p$, $\boldsymbol{b} \in \mathbb{R}^m$) and $\boldsymbol{c} := f(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^m$. Consider a function $\boldsymbol{\varphi} : W \cap \mathbb{R}^m \to \mathbb{R}^m$ defined by $\boldsymbol{y} \mapsto \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{y})$. Its Jacobi matrix is

$$J(\boldsymbol{a}, \boldsymbol{y}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{a}, \boldsymbol{y}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1}(\boldsymbol{a}, \boldsymbol{y}) & \dots & \dfrac{\partial f_1}{\partial y_m}(\boldsymbol{a}, \boldsymbol{y}) \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial y_1}(\boldsymbol{a}, \boldsymbol{y}) & \dots & \dfrac{\partial f_m}{\partial y_m}(\boldsymbol{a}, \boldsymbol{y}) \end{pmatrix}$$

Assume that $J(\boldsymbol{a}, \boldsymbol{y})$ is invertible at $\boldsymbol{y} = \boldsymbol{b}$. Then there exist open sets $U \subset \mathbb{R}^p$, $\boldsymbol{a} \in U$, and $V \subset \mathbb{R}^m$, $\boldsymbol{b} \in V$, and a smooth map $\boldsymbol{g} : U \to V$ with $\boldsymbol{g}(\boldsymbol{a}) = \boldsymbol{b}$ such that

$$\{(\boldsymbol{x}, \boldsymbol{y}) \in U \times V \,|\, \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}\} = \{(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) \,|\, \boldsymbol{x} \in U\}$$

(i.e. the level set of points $(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}$ is locally a *graph* of some smooth function $\boldsymbol{g} : U \to V$).

We will use this theorem in a particular case of $m = 1$: having a function

$$f : \mathbb{R}^{p+1} \to \mathbb{R}, \quad (x_1, \dots, x_p, y) \mapsto f(\boldsymbol{x}, y), \quad f(\boldsymbol{x}_0, y_0) = c$$

with $\dfrac{\partial f}{\partial y}(\boldsymbol{x}_0, y_0) \neq 0$, one has $y = g(\boldsymbol{x})$ in a neighborhood of $\boldsymbol{x}_0$ for $f(\boldsymbol{x}, y) = c$.