Durham University
Pavel Tumarkin

Michaelmas 2016

Differential Geometry III, Term 1 (Section 7)

7 Tangent plane, first fundamental form and area

7.1 The tangent plane

Definition 7.1. Let S be a regular surface and $p \in S$. A *tangent vector* to S at p is the tangent vector $\alpha'(0) \in \mathbb{R}^3$ of a smooth (not necessarily regular) curve $\alpha\colon (-\varepsilon, \varepsilon) \longrightarrow S \subset \mathbb{R}^3$ with $\alpha(0) = p$ (for some $\varepsilon > 0$).

Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of S, $\boldsymbol{q} \in U$, $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{p}$. Recall that the differential (or derivative) $d_{\boldsymbol{q}}\boldsymbol{x}$ is a linear map $d_{\boldsymbol{q}}\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$. By the definition of a regular surface, $d_{\boldsymbol{q}}\boldsymbol{x}$ has full rank at every point, so the dimension of the image is equal to 2.

Definition 7.2. The plane $d_{\boldsymbol{q}}\boldsymbol{x}(\mathbb{R}^2)$ is called the *tangent plane* to S at $\boldsymbol{p}$ and is denoted by $T_{\boldsymbol{p}}S$.

Proposition 7.3. Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of a regular surface S with $U \subset \mathbb{R}^2$ open, and let $\boldsymbol{q} \in U$. Then Then the tangent plane $T_{\boldsymbol{p}}S$ coincides with the set of all tangent vectors to S at $\boldsymbol{p}$.

Remark 7.4. (a) Since the definition of a tangent vector does not depend on a parametrization, Prop. 7.3 implies that the tangent plane does not depend on a parametrization either.

(b) If $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ and $\boldsymbol{w} = \alpha'(0)$, then w has coordinates $(u'(0), v'(0))$ with respect to the basis $\{\boldsymbol{x}_u(\boldsymbol{q}), \boldsymbol{x}_v(\boldsymbol{q})\}$.

Example 7.5.

(a) **Tangent plane to graph of a function:** Let $g\colon U \longrightarrow \mathbb{R}$ be a smooth function on an open subset U of $\mathbb{R}^2$, i.e.

$$S := \operatorname{graph} g = \{\, (u, v, g(u, v)) \,|\, (u, v) \in U \,\}$$

is a regular surface with parametrisation $\boldsymbol{x}(u, v) := (u, v, g(u, v))$. Then the tangent plane $T_{\boldsymbol{p}}S$ to S at $\boldsymbol{p} = (u, v, g(u, v))$ is generated by

$$\{\boldsymbol{x}_u(\boldsymbol{q}), \boldsymbol{x}_v(\boldsymbol{q})\} = \{(1, 0, g_u(u, v)), (0, 1, g_v(u, v))\},$$

where $\boldsymbol{q} = (u, v)$.

(b) **Tangent plane to a level set of a function:** Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function, and let $c \in \mathbb{R}$ be a regular value of f (i.e., $\nabla f(\boldsymbol{p}) \neq \boldsymbol{0}$ for all $\boldsymbol{p} \in \mathbb{R}^3$ with $f(\boldsymbol{p}) = c$). We have seen that $S := f^{-1}(c)$ is a regular surface.

Lemma 7.6. Let $\boldsymbol{p} \in S$, then $T_{\boldsymbol{p}}S$ is the plane in $\mathbb{R}^3$ *orthogonal* to $\nabla f(\boldsymbol{p})$.

7.2 The first fundamental form

Let $\boldsymbol{p} \in S$. We can consider the restriction of the inner product $(\cdot)\colon \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$, $(\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} \cdot \boldsymbol{w}$, to $T_{\boldsymbol{p}}S \subset \mathbb{R}^3$. We denote the restriction by $\langle \cdot, \cdot \rangle_{\boldsymbol{p}}$, i.e.,

$$\langle \cdot, \cdot \rangle_{\boldsymbol{p}} \colon T_{\boldsymbol{p}}S \times T_{\boldsymbol{p}}S \longrightarrow \mathbb{R}, \qquad (\boldsymbol{w}_1, \boldsymbol{w}_2) \mapsto \boldsymbol{w}_1 \cdot \boldsymbol{w}_2.$$

This map is

- *bilinear*, i.e, linear in both of its arguments;
- *symmetric*, i.e., $\langle \boldsymbol{w}_2, \boldsymbol{w}_1 \rangle_{\boldsymbol{p}} = \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle_{\boldsymbol{p}}$ for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_{\boldsymbol{p}}S$;
- and *positive*, i.e., $\|\boldsymbol{w}\|_{\boldsymbol{p}}^2 := \langle \boldsymbol{w}, \boldsymbol{w} \rangle \geq 0$ and $\|\boldsymbol{w}\|_{\boldsymbol{p}}^2 = 0$ implies $\boldsymbol{w} = 0$ for all $\boldsymbol{w} \in T_{\boldsymbol{p}}S$.

We can now measure the length of a tangent vector $\boldsymbol{w} \in T_{\boldsymbol{p}}S$ and the angle between two tangent vectors $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_{\boldsymbol{p}}S$ by

$$\sqrt{\langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{p}}} \qquad \text{and} \qquad \cos\vartheta = \frac{\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle_{\boldsymbol{p}}}{\sqrt{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle_{\boldsymbol{p}}}\sqrt{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle_{\boldsymbol{p}}}}.$$

A quadratic form $I_{\boldsymbol{p}}$ is obtained from a bilinear form $\langle \cdot, \cdot \rangle_{\boldsymbol{p}}$ by setting $I_{\boldsymbol{p}}(\boldsymbol{w}) := \langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{p}}$.

Definition 7.7. The quadratic form $I_{\boldsymbol{p}}\colon T_{\boldsymbol{p}}S \longrightarrow \mathbb{R}$, $I_{\boldsymbol{p}}(\boldsymbol{w}) := \langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{p}} = \|\boldsymbol{w}\|^2$ is called the *first fundamental form* at $\boldsymbol{p} \in S$.

Definition 7.8. The functions $E, F, G\colon U \longrightarrow \mathbb{R}$ defined by

$$E := \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle_{\boldsymbol{p}}, \quad F := \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle_{\boldsymbol{p}}, \quad G := \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle_{\boldsymbol{p}}$$

are called the *coefficients* of the first fundamental form in the local parametrization $\boldsymbol{x}\colon U \longrightarrow S$.

Note that the coefficients of the first fundamental form depend on the parametrisation $\boldsymbol{x}$!

Remark 7.9. If $(a, b) \in \mathbb{R}^2$ are the coordinates of a vector $\boldsymbol{w} \in T_{\boldsymbol{p}}S$ with respect to the basis $\{\boldsymbol{x}_u(\boldsymbol{q}), \boldsymbol{x}_v(\boldsymbol{q})\}$, then

$$I_{\boldsymbol{p}}(\boldsymbol{w}) = a^2E + 2abF + b^2G = \begin{pmatrix} a & b \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

Since $I_{\boldsymbol{p}}$ is positive ($I_{\boldsymbol{p}}(\boldsymbol{w}) = \|\boldsymbol{w}\|^2 \geq 0$ and $I_{\boldsymbol{p}}(\boldsymbol{w}) = 0$ implies $\boldsymbol{w} = \boldsymbol{0}$), we have

$$E > 0, \quad G > 0 \quad \text{and} \quad \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} = EG - F^2 > 0.$$

Example 7.10. Let S be a plane in $\mathbb{R}^3$ given by an equation $ax + by + cz + d = 0$, and assume without loos of generality that $c \neq 0$. Then

$$\boldsymbol{x}_x(x, y) = (1, 0, -a/c) \qquad \text{and} \qquad \boldsymbol{x}_y(x, y) = (0, 1, -b/c).$$

In particular, we have

$$E(x, y) = 1 + \frac{a^2}{c^2}, \qquad F(x, y) = \frac{ab}{c^2}, \qquad G(x, y) = 1 + \frac{b^2}{c^2}$$

Example 7.11. Coefficients of the first fundamental form for a graph of a function: Let a surface be given by a graph of a function g, namely $\boldsymbol{x}(u,v) := (u,v,g(u,v)) = (u,v,u^2+v^2)$ for $(u,v) \in U := \mathbb{R}^2$. Then

$$\boldsymbol{x}_u(u,v) = (1,0,g_u) = (1,0,2u) \qquad \text{and} \qquad \boldsymbol{x}_v(u,v) = (0,1,g_v) = (0,1,2v).$$

In particular, we have

$$\begin{aligned} E &= (1,0,g_u)\cdot(1,0,g_u) = 1+g_u^2, &&\text{here} \quad E(u,v) = 1+4u^2,\\ F &= (1,0,g_u)\cdot(0,1,g_v) = g_u g_v, &&\text{here} \quad F(u,v) = 8uv,\\ G &= (0,1,g_v)\cdot(0,1,g_v) = 1+g_v^2 &&\text{here} \quad G(u,v) = 1+4v^2, \end{aligned}$$

Example 7.12. Coefficients of the first fundamental form for a surface of revolution: Let S be obtained by rotating the space curve given by $\boldsymbol{\alpha}(v) = (f(v),0,g(v))$, $v \in \mathbb{R}$, around the z-axis (without self-intersections and without meeting the z-axis, i.e., $f(v) = 0$). A parametrization is then given by

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

$(u,v) \in (-\pi,\pi)\times\mathbb{R}$. Here, we have

$$\boldsymbol{x}_u(u,v) = (-f(v)\sin u, f(v)\cos u, 0) \quad \text{and} \quad \boldsymbol{x}_v(u,v) = (f'(v)\cos u, f'(v)\sin u, g'(v)).$$

The coefficients of the first fundamental form in this parametrization are

$$E(u,v) = f(v)^2, \quad F(u,v) = 0 \quad \text{and} \quad G(u,v) = |f'(v)|^2 + |g'(v)|^2 = \|\boldsymbol{\alpha}'(v)\|^2.$$

7.3 Arc lengths of a curve and angles between curves in a surface

The aim of the following remark is to calculate the arc length of a curve in a surface *using only the coefficients of the first fundamental form.*

Definition 7.13. Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a curve on a regular surface S. Then the length of $\boldsymbol{\alpha}$, measured from a point $\boldsymbol{\alpha}(u_0)$ for some $u_0 \in I$, is

$$\ell(u) := \int_{u_0}^{u} \sqrt{\langle \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle_{\boldsymbol{\alpha}(s)}}\,\mathrm{d}s.$$

Proposition 7.14 (evident)**.**

$$\ell(u) := \int_{u_0}^{u} [I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s))]^{1/2}\,\mathrm{d}s.$$

Remark 7.15. Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a curve on a regular surface S and $\boldsymbol{x}\colon U \longrightarrow S$ a local parametrization such that $\boldsymbol{\alpha}(I) \subset \boldsymbol{x}(U)$. Denote by $\boldsymbol{\beta} = (u,v)$ the corresponding curve in the parameter domain (i.e., $\boldsymbol{\alpha}(s) = \boldsymbol{x}(\boldsymbol{\beta}(s)) = \boldsymbol{x}(u(s),v(s))$).

Let E,F,G be the coefficients of the first fundamental form w.r.t. the parametrization $\boldsymbol{x}$. Then the arc lengths of $\boldsymbol{\alpha}$ from $s_0 \in I$ to $s_1 \in I$ can be expressed in terms of E,F,G only as follows:

$$\ell(s_1) = \int_{s_0}^{s_1} [I_{\boldsymbol{\alpha}(t)}(\boldsymbol{\alpha}'(t))]^{1/2}\,\mathrm{d}t = \int_{s_0}^{s_1} \sqrt{u'(t)^2 E(\boldsymbol{\beta}(t)) + 2u'(t)v'(t)F(\boldsymbol{\beta}(t)) + v'(t)^2 G(\boldsymbol{\beta}(t))}\,\mathrm{d}t.$$

Example 7.16. The hyperbolic plane. We construct a surface by fixing the coefficients of the first fundamental form E, F, G only. Actually, this is the first example which cannot (in total) be realized as a surface in $\mathbb{R}^3$.

Let $U := \{ (u, v) \in \mathbb{R}^2 \mid v > 0 \}$ be the upper halfplane and set

$$E(u,v) := \frac{1}{v^2}, \quad F(u,v) := 0 \quad \text{and} \quad G(u,v) := \frac{1}{v^2},$$

i.e., $F = 0$ and $E = G$.

Let us now assume that there is a surface S in an ambient space $\mathbb{R}^n$ and a parametrization $\boldsymbol{x}\colon U \longrightarrow S$ such that the corresponding coefficients of the fundamental form have the desired form.

Consider a curve $\boldsymbol{\alpha}\colon (0,\infty) \longrightarrow S$ given by $\boldsymbol{\alpha}(s) = \boldsymbol{x}(0,s)$. In the coordinates on U, the curve has the form $\boldsymbol{\beta}\colon (0,\infty) \longrightarrow U$, $\boldsymbol{\beta}(s) = (0,s)$. Then

$$\|\boldsymbol{\alpha}'(s)\|^2 = 0E(0,s) + 0 + 1G(0,s) = \frac{1}{s^2}$$

Therefore, the arc length of $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}(a)$ to $\boldsymbol{\alpha}(b)$ on S is

$$\int_a^b \|\boldsymbol{\alpha}'(s)\| \, \mathrm{d}s = \int_a^b \frac{1}{s} \, \mathrm{d}s = \log b - \log a = \log \frac{b}{a}.$$

The upper half-plane $U = \mathbb{R} \times (0,\infty)$ together with the first fundamental form above is called the *upper half-plane model of the hyperbolic plane*. The corresponding surface S, the *hyperbolic plane*, is sometimes denoted by $\mathbb{H}$.

Remark. Coordinate curves and angle. Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrization of a regular surface $S \subset \mathbb{R}^n$, $(u_0, v_0) \in U$. Consider the curves

$$\boldsymbol{\alpha}_1(s) = \boldsymbol{x}(u_0 + s, v_0) \qquad \text{and} \qquad \boldsymbol{\alpha}_2(s) = \boldsymbol{x}(u_0, v_0 + s)$$

with s being small. These curves are called the *coordinate curves* of the parametrization $\boldsymbol{x}$. The angle formed by the two curves meeting in (u_0, v_0) can be calculated by

$$\cos\vartheta = \frac{\boldsymbol{\alpha}_1'(0) \cdot \boldsymbol{\alpha}_2'(0)}{\|\boldsymbol{\alpha}_1'(0)\| \|\boldsymbol{\alpha}_2'(0)\|}.$$

But $\boldsymbol{\alpha}_1'(0) = \boldsymbol{x}_u(u_0, v_0)$ and $\boldsymbol{\alpha}_2'(0) = \boldsymbol{x}_v(u_0, v_0)$, so that (omitting the argument (u_0, v_0))

$$\cos\vartheta = \frac{\boldsymbol{x}_u \cdot \boldsymbol{x}_v}{\|\boldsymbol{x}_u\| \|\boldsymbol{x}_v\|} = \frac{F}{\sqrt{EG}}.$$

7.4 Area of subsets of a surface

Definition 7.17. Let $R_0 \subset U$, $R = \boldsymbol{x}(R_0) \subset S$. The *area* of a region $R = \boldsymbol{x}(R_0)$ is defined as

$$\operatorname{area}(R) := \int_{R_0} \sqrt{EG - F^2} \, \mathrm{d}u \, \mathrm{d}v.$$

Example 7.18. Let S be a half of a cylinder parametrized by

$$\boldsymbol{x}(u,v) = (u, v, \sqrt{1 - v^2}), \qquad (u,v) \in U = (-1,1) \times (-1,1)$$

Then $E \equiv 1$, $F \equiv 0$, $G = 1/(1 - v^2)$, so

$$\operatorname{area}(S) = \int_U \sqrt{EG - F^2} \, \mathrm{d}u \, \mathrm{d}v = \int_{-1}^1 \mathrm{d}u \int_{-1}^1 \sqrt{1/(1 - v^2)} \, \mathrm{d}v = 2\pi$$

The definition of area depends at first sight on the local parametrization $\boldsymbol{x}\colon U \longrightarrow S$. Actually, it does not:

Proposition 7.19. Assume that we have two local parametrizations $\boldsymbol{x}_1\colon U_1 \longrightarrow S$ and $\boldsymbol{x}_2\colon U_2 \longrightarrow S$ with $\boldsymbol{x}_1(U_1) = \boldsymbol{x}_2(U_2) =: W$. Denote by E_1, F_1, G_1 and E_2, F_2, G_2 the coefficients of the first fundamental form in the parametrisation $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively.

Let $R \subset W$. Denote by $R_1 := \boldsymbol{x}_1^{-1}(R)$ and $R_2 := \boldsymbol{x}_2^{-1}(R)$ the corresponding regions in the respective parameter domains. Then

$$\int_{R_1} \sqrt{E_1 G_1 - F_1^2}\, \mathrm{d}u_1\, \mathrm{d}v_1 = \int_{R_2} \sqrt{E_2 G_2 - F_2^2}\, \mathrm{d}u_2\, \mathrm{d}v_2.$$

Example 7.20.

(a) **The sphere.** Let S be the sphere of radius $r > 0$ in $\mathbb{R}^3$,

$$\boldsymbol{x}(u, v) = (r \cos u \sin v, r \sin u \sin v, r \cos v)$$

(v measures *latitude*, u measures *longitude*, and (u, v) are called *spherical coordinates*). We have

$$E(u, v) = r^2 \sin^2 v, \quad F(u, v) = 0 \quad \text{and} \quad G(u, v) = r^2,$$

so that $EG - F^2 = r^4 \sin^2 v$.

Let us compute the area of a “slice” of the sphere enclosed by planes $z = z_0$ and $z = z_1$, where $-r \le z_1 < z_0 \le r$. This corresponds to the domain $\arccos z_0 \le v \le \arccos z_1, u \in (0, 2\pi)$. Therefore the area is

$$\int_0^{2\pi} \mathrm{d}u \int_{\arccos z_0}^{\arccos z_1} r^2 \sin^2 v\, \mathrm{d}v = 2\pi r^2 (z_0 - z_1).$$

(b) **Torus of revolution:** Consider the parametrization

$$\begin{aligned} \boldsymbol{x}\colon U := (0, 2\pi) \times (0, 2\pi) &\longrightarrow S, \\ \boldsymbol{x}(u, v) := \big((R + r\cos v)\cos u,& (R + r\cos v)\sin u, r \sin v\big) \end{aligned}$$

for $0 < r < R$. This surface is a surface of revolution, obtained by rotating the curve $\boldsymbol{\alpha}$ given by

$$\boldsymbol{\alpha}(v) = \big((R + r\cos v), 0, r \sin v\big)$$

(which is a circle of radius r in the (x, z)-plane centered at the point $(R, 0, 0)$) around the z-axis. Then

$$\begin{aligned} \boldsymbol{x}_u(u, v) &= \big(-(R + r\cos v)\sin u, (R + r\cos v)\cos u, 0\big), \\ \boldsymbol{x}_v(u, v) &= \big(-r \sin v \cos u, -r \sin v \sin u, r \cos v\big) \end{aligned}$$

and therefore

$$E(u, v) = (R + r\cos v)^2, \quad F(u, v) = 0 \quad \text{and} \quad G(u, v) = r^2.$$

In particular, $\sqrt{EG - F^2} = (R + r\cos v)r$, hence

$$\operatorname{area}(S) = \int_0^{2\pi} \int_0^{2\pi} (R + r\cos v) r\, \mathrm{d}u\, \mathrm{d}v = 4\pi^2 r R.$$

(c) **Hyperbolic plane:** Recall that we have the parameter domain $U := \mathbb{R} \times (0, \infty)$ together with the coefficients of the fundamental form

$$E(u,v) = G(u,v) = \frac{1}{v^2}, \quad F(u,v) = 0,$$

and $\sqrt{EG - F}(u,v) = 1/v^2$. Let $R_{a,b} := (0,b) \times (a, 2a)$, then the corresponding region in the hyperbolic plane $\mathbb{H}$ has area

$$\text{area}(R) = \int_{R_{a,b}} \frac{1}{v^2} \, \mathrm{d}u \, \mathrm{d}v = \int_0^b \mathrm{d}u \int_a^{2a} \frac{1}{v^2} \, \mathrm{d}v = b/2a.$$

In particular, if $b = a$, we obtain $1/2$ which does not depend on a.