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Differential Geometry III, Term 2 (Section 8)

8 Smooth maps between surfaces

Recall that $f\colon U \longrightarrow \mathbb{R}^m$ is smooth at $p \in U$ if all partial derivatives of f at p exist and are continuous. We need $U \subset \mathbb{R}^n$ to be *open* to be able to define a partial derivative.

Let $S \subset \mathbb{R}^n$ be a regular surface and $f\colon S \longrightarrow \mathbb{R}^m$. Since S is not open in $\mathbb{R}^n$ ($n \geq 3$), we need to define smoothness of f on S.

Definition 8.1. We say that $f\colon S \longrightarrow \mathbb{R}^m$ is smooth at p if

$$f \circ \boldsymbol{x}\colon U \longrightarrow \mathbb{R}^m$$

is smooth at q where $\boldsymbol{x}\colon U \longrightarrow S$ is a parametrization with $\boldsymbol{x}(q) = p$.

Remark 8.2. This definition does not depend on the parametrization $\boldsymbol{x}$. Indeed, if $\boldsymbol{y}\colon V \longrightarrow S$ is another parametrization (assume that $\boldsymbol{x}(U) = \boldsymbol{y}(V)$), then there exists a diffeomorphism $h\colon U \longrightarrow V$ such that $\boldsymbol{y} = \boldsymbol{x} \circ h$ (change of parameter). In particular, $f \circ \boldsymbol{y} = (f \circ \boldsymbol{x}) \circ h$ is also smooth.

8.1 The Gauss map

Let S be a regular surface in $\mathbb{R}^3$.

Definition 8.3. The *Gauss map*

$$\boldsymbol{N}\colon S \longrightarrow S^2$$

assigns, to each point $p \in S$, the unit normal to S at p, i.e., the unit vector orthogonal to $T_pS \subset \mathbb{R}^3$ (which is determined up to sign only!). Here, $S^2 := \{\, (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \,\}$ is the unit sphere in $\mathbb{R}^3$.

In a local parametrization $\boldsymbol{x}\colon U \longrightarrow S$ of S, we have

$$\boldsymbol{N} \circ \boldsymbol{x}(u,v) := \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}(u,v),$$

and this map is always smooth.

Example 8.4.

(a) **Plane in $\mathbb{R}^3$:** $S = \{\, (x,y,z)\mathbb{R}^3 \mid ax + by + cz + d = 0 \,\}$. Then $\boldsymbol{N} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} \equiv$ const.

(b) **Graph of a function:** $S = \{\, (u, v, g(u,v)) \mid (u,v) \in U \,\}$, $g\colon U \longrightarrow \mathbb{R}$ smooth, then $\boldsymbol{x}_u = (1, 0, g_u)$, $\boldsymbol{x}_v = (0, 1, g_v)$, then the Gauss map is given by $\boldsymbol{N}\colon S \longrightarrow S^2$

$$\boldsymbol{N} \circ \boldsymbol{x} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|} = \frac{1}{\sqrt{1 + (g_u)^2 + (g_v)^2}}(-g_u, -g_v, 1).$$

As an example, take $g(u,v) = u^2 + v^2$, then

$$\boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{1}{\sqrt{1 + 4u^2 + 4v^2}}(-2u, -2v, 1)$$

Also, $S = f^{-1}(0)$ for $f(x,y,z) = x^2 + y^2 - z$, so $\nabla f = (2x, 2y, -1)$ is proportional to $\boldsymbol{N}$ as expected.

(c) **The catenoid:** $\boldsymbol{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v)$, then

$$\boldsymbol{x}_u(u,v) = (-\cosh v \sin u, \cosh v \cos u, 0) \quad \text{and} \quad \boldsymbol{x}_v(u,v) = (\sinh v \cos u, \sinh v \sin u, 1)$$

so that

$$(\boldsymbol{x}_u \times \boldsymbol{x}_v)(u,v) = (\cosh v \cos u, \cosh v \sin u, -\cosh v \sinh v),$$

and therefore

$$\boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v).$$

(d) **The sphere:** $\boldsymbol{N}\colon S^2 \longrightarrow S^2$ is given by $\boldsymbol{N}(p) = p$.

Remark. The Gauss map is well defined on $\boldsymbol{x}(U)$, but we may not be able to define it (continuously) on all S

Example 8.5. Möbius band

Definition 8.6. A surface in $\mathbb{R}^3$ is *non-orientable* if it is not possible to define the Gauss map globally.

Example 8.7. Further maps on surfaces. Let $S \subset \mathbb{R}^3$ be a surface.

(a) **Height function.** Fix $\boldsymbol{v} \in S^2$, and define a function $h\colon S \longrightarrow \mathbb{R}$ by $h(p) := p \cdot \boldsymbol{v}$. Then h is smooth. You can think of h measuring the height of S if you stand on the plane orthogonal to $\boldsymbol{v}$ fixed e.g. at the origin of $\mathbb{R}^3$.

(b) **Distance squared function.** Let $a \in \mathbb{R}^3$ and define $d^2\colon S \longrightarrow \mathbb{R}$ by $d^2(p) := \|p - a\|^2 = (p-a) \cdot (p-a)$, then d^2 is smooth. (d measures the distance of p from a *in the ambient space* $\mathbb{R}^3$).

8.2 The derivative of a smooth map between surfaces

Definition 8.8. Let S be a regular surface in $\mathbb{R}^\ell$, $p \in S$ and $f\colon S \longrightarrow \mathbb{R}^m$ a smooth map. The *derivative of f at p* is a linear map

$$d_p f\colon T_p S \longrightarrow \mathbb{R}^m$$

such that

$$d_p f(\boldsymbol{x}_u) = \partial_u (f \circ \boldsymbol{x})(q) \quad \text{and} \quad d_p f(\boldsymbol{x}_v) = \partial_v (f \circ \boldsymbol{x})(q)$$

for a local parametrization $\boldsymbol{x}\colon U \longrightarrow S$ of S with $\boldsymbol{x}(q) = p$, $q \in U \subset \mathbb{R}^2$. For short, we write

$$\boldsymbol{f}_u := d_p f(\boldsymbol{x}_u) \quad \text{and} \quad \boldsymbol{f}_v := d_p f(\boldsymbol{x}_v),$$

suppressing the local parametrisation $\boldsymbol{x}$ in the notation $\boldsymbol{f}_u$ and $\boldsymbol{f}_v$.

Remark 8.9.

(a) As $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ is a basis of $T_p S$, and $\boldsymbol{w} \in T_p$ can be written as $\boldsymbol{w} = a\boldsymbol{x}_u + \boldsymbol{x}_v$, we have

$$d_p f(\boldsymbol{w}) = d_p f(a\boldsymbol{x}_u + b\boldsymbol{x}_v) = a d_p f(\boldsymbol{x}_u) + b d_p f(\boldsymbol{x}_v)$$

by the linearity of $d_p f$.

(b) $d_p f$ does not depend on the choice of local parametrization $\boldsymbol{x}$. Indeed, if we take $\boldsymbol{w} \in T_p S$ and compute its image, then if $\boldsymbol{w} = \boldsymbol{\alpha}'(0)$ for $\boldsymbol{\alpha}\colon I \longrightarrow S$ a smooth curve, $\boldsymbol{\alpha}(0) = p$, we have $d_p f(\boldsymbol{w}) = (f \circ \alpha)'(0))$.

Example 8.10. (a) Let $S = \{\, (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1 \,\}$ be a cylinder in $\mathbb{R}^3$ and $f\colon S \longrightarrow \mathbb{R}$ be given by $f(p) = p \cdot p = \|p\|^2$. A local parametrization of S is given by

$$\boldsymbol{x}\colon U \longrightarrow S, \qquad \boldsymbol{x}(\vartheta, z) = (\cos\vartheta, \sin\vartheta, z), \qquad (\vartheta, r) \in U$$

Here, at least two parameter domains $U_1 = (0, 2\pi) \times \mathbb{R}$ and $U_2 = (-\pi, \pi) \times \mathbb{R}$ are needed in order to cover the entire cylinder. Then we have $(f \circ \boldsymbol{x})(\vartheta, z) = f(\cos\vartheta, \sin\vartheta, z)$ and

$$d_p f(\boldsymbol{x}_\vartheta) = \boldsymbol{f}_\vartheta = \frac{\partial}{\partial\vartheta}(f \circ \boldsymbol{x}) = 0 \quad \text{and} \quad d_p f(\boldsymbol{x}_z) = \boldsymbol{f}_z = \frac{\partial}{\partial z}(f \circ \boldsymbol{x}) = 2z.$$

(b) **(Gauss map of a catenoid)** Let S be parametrized by

$$\boldsymbol{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v),$$

then its Gauss map is given by

$$\boldsymbol{N}(\boldsymbol{x}(u, v)) = \frac{1}{\cosh v}\big(\cos u, \sin u, -\sinh v\big).$$

In particular, the derivative is

$$\begin{aligned} d_p\boldsymbol{N}(\boldsymbol{x}_u) &= \boldsymbol{N}_u = \frac{1}{\cosh v}\big(-\sin u, \cos u, 0\big) \quad \text{and} \\ d_p\boldsymbol{N}(\boldsymbol{x}_v) &= \boldsymbol{N}_v = \frac{1}{\cosh^2 v}\big(-\cos u \sinh v, -\sin u \sinh v, -1\big). \end{aligned}$$

Proposition 8.11 (Chain Rule). Let $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ be smooth maps between the surfaces S_1, S_2 and S_3, then $g \circ f\colon S_1 \longrightarrow S_3$ is smooth and its derivative is given by

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f\colon T_pS_1 \longrightarrow T_{g(f(p))}S_3$$

as linear maps, or pointwise,

$$d_p(g \circ f)(\boldsymbol{w}) = d_{f(p)}g(d_p f(\boldsymbol{w}))$$

for all $\boldsymbol{w} \in T_pS_1$ and $p \in S_1$.

8.3 Isometries and conformal maps

Let $S \subset \mathbb{R}^\ell$ be a regular surface. Recall that the *first fundamental form* (1^{st}FF) is given by

$$I_p\colon T_pS \longrightarrow \mathbb{R}, \qquad I_p(\boldsymbol{w}) = \langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\mathbb{R}^\ell} = \|\boldsymbol{w}\|^2_{\mathbb{R}^\ell}.$$

Recall also that the 1^{st}FF is needed to calculate

- lengths of curves in S,
- angles between curves in S and
- the area of subsets of S.

Let now S and $\widetilde{S}$ be two surfaces with 1^{st}FFs I and $\widetilde{I}$, respectively, let $f\colon S \longrightarrow \widetilde{S}$ be a smooth map. If $d_p f\colon T_pS \longrightarrow T_{f(p)}\widetilde{S}$ "preserves" I_p and $\widetilde{I}_{f(p)}$, then these calculations should give the same result, i.e., S and $\widetilde{S}$ are bassically the same from a metric point of view (at least locally: see Example 8.13 (a) below)

Definition 8.12. Let $f\colon S \longrightarrow \widetilde{S}$ be a smooth map between two surfaces S and $\widetilde{S}$.

(a) The map f is called a *(local) isometry* if

$$\langle d_pf(\boldsymbol{w}_1), d_pf(\boldsymbol{w}_2)\rangle_{f(p)} = \langle \boldsymbol{w}_1, \boldsymbol{w}_2\rangle_p$$

for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_pS$ and $p \in S$. The surfaces S and $\widetilde{S}$ are called *(locally) isometric* if there is a (local) isometry between them.

(b) The map f is called a *(global) isometry* if f is a local isometry and, additionally, $f\colon S \longrightarrow \widetilde{S}$ is *bijective*.

The surfaces S and $\widetilde{S}$ are called *(globally) isometric* if there is a (globally) isometry between them.

(c) The map f is called *conformal* if there is a smooth function

$$\lambda\colon S \longrightarrow (0, \infty)$$

such that

$$\langle d_pf(\boldsymbol{w}_1), d_pf(\boldsymbol{w}_2)\rangle_{f(p)} = \lambda(p)\langle \boldsymbol{w}_1, \boldsymbol{w}_2\rangle_p$$

for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_pS$ and $p \in S$.

The surfaces S and $\widetilde{S}$ are called *conformally equivalent* if there is a conformal map between them.

Remark.

(a) Given a symmetric bilinear form $\langle , \rangle$, one can write $\langle \boldsymbol{w}_1, \boldsymbol{w}_2\rangle = \frac{1}{2}(\|\boldsymbol{w}_1 + \boldsymbol{w}_2\|^2 - \|\boldsymbol{w}_1\|^2 - \|\boldsymbol{w}_2\|^2)$, which means that being a local isometry is equivalent to preserving 1^{st}FF, i.e. $\widetilde{I}_{f(p)}(d_pf(\boldsymbol{w})) = I_p(\boldsymbol{w})$, cf. Prop. 8.15.

(b) A conformal map with $\lambda \equiv 1$ is obviously a local isometry.

(c) A global isometry is obviously a local isometry, but not vice versa (see Example 8.13 (c) below).

(d) Conformal maps preserve angles. Indeed,

$$\vartheta = \angle(\boldsymbol{w}_1, \boldsymbol{w}_2) := \frac{\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle_p}{\|\boldsymbol{w}_1\|_p\|\boldsymbol{w}_1\|_p} \quad \text{and}$$

$$\angle(d_pf(\boldsymbol{w}_1), d_pf(\boldsymbol{w}_2)) := \frac{\langle d_pf(\boldsymbol{w}_1), d_pf(\boldsymbol{w}_1)\rangle_p}{\|d_pf(\boldsymbol{w}_1)\|_p\|d_pf(\boldsymbol{w}_1\|_p)} = \frac{\lambda(p)\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle_p}{\sqrt{\lambda(p)}\|\boldsymbol{w}_1\|_p\sqrt{\lambda(p)}\|\boldsymbol{w}_1\|_p} = \vartheta$$

since the factors involving $\lambda(p) > 0$ cancel each other.

(e) Local isometries preserve lengths of curves (but not distances between points). Global isometries preserve distances.

Example 8.13.

(a) Let $S = (0, 2\pi) \times \mathbb{R}$ and $\widetilde{S} = \{\,(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\,\}$ (a cylinder). Define $f\colon S \longrightarrow \widetilde{S}$ by $f(\vartheta, z) = (\cos\vartheta, \sin\vartheta, z)$ for $p = (\vartheta, z) \in S$. We can think of S as being parametrized by itself (as a subset of the plane $\mathbb{R}^2$), and $T_pS = \mathbb{R}^2$.

One way to show that f is a local isometry is to ensure that it preserves 1^{st}FF (the identity matrix), which is an elementary computation of $\boldsymbol{f}_\vartheta$ and $\boldsymbol{f}_z$ and their dot products, cf. Prop. 8.15.

Alternatively, one can compute the differential of f explicitely. Write $\boldsymbol{w} = (a, b) \in T_pS$. We need $\boldsymbol{\alpha}\colon I \longrightarrow S$ with I being an open interval containing 0, $\boldsymbol{\alpha}(0) = p$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$. Take a line through $p \in S \subset \mathbb{R}^2$ in direction $\boldsymbol{w}$, i.e.

$$\boldsymbol{\alpha}(t) = p + t\boldsymbol{w} = (\vartheta + ta, z + tb).$$

Then

$$d_pf(\boldsymbol{w}) = d_pf(\boldsymbol{\alpha}'(0)) = (f \circ \boldsymbol{\alpha})'(0)$$

Here, we have

$$(f \circ \boldsymbol{\alpha})(t) = \big(\cos(\vartheta ta), \sin(\vartheta + ta), z + tb\big),$$

so that

$$(f \circ \boldsymbol{\alpha})'(0) = (-a \sin\vartheta, a\cos\vartheta, b) = d_pf(\boldsymbol{w}).$$

Now,

$$\langle d_pf(\boldsymbol{w}), d_pf(\boldsymbol{w})\rangle_{f(p)} = \langle(-a\sin\vartheta, a\cos\vartheta, b), (-a\sin\vartheta, a\cos\vartheta, b)\rangle = a^2 + b^2,$$

but we also have $\langle \boldsymbol{w}, \boldsymbol{w}\rangle_p = a^2 + b^2$, hence f is a local isometry.

(b) If we consider $f\colon S \longrightarrow \{\, (x, y, z) \mid x^2 + y^2 = 1, (x, y) \neq (1, 0)\,\}$, then f is bijective (check this!) and f is indeed a *global* isometry.

(c) If we consider $f\colon \mathbb{R} \times \mathbb{R} \longrightarrow \widetilde{S}$ (with the same definition of $f(\vartheta, z)$ as before, but now $\vartheta \in \mathbb{R}$), then f is still a local isometry (the calculation remains the same as above), but not a *global* isometry: f is no longer injective and hence not bijective.

Example 8.14 (Conformal bijections of $\mathbb{R}^2$). As one can recall from Complex Analysis, conformal maps are holomorphic (or anti-holomorphic) and vise versa. Thus conformal bijections of the plane are holomorphic one-to-one maps. They must have a single pole at infinity, so they are polynomial of degree one (possibly with conjugation), i.e. $f(z) = az + b$ or $f(z) = a\bar{z} + b$, $a, b \in \mathbb{C}$, $a \neq 0$. The conformal factor is $\lambda(z) = |a|^2$.

Proposition 8.15. Let S, $\widetilde{S}$ be two surfaces and $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of S.
A map $f\colon S \longrightarrow \widetilde{S}$ is a local isometry on $\boldsymbol{x}(U)$ if and only if

$$\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle = E, \quad \langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle = F \quad \text{and} \quad \langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle = G, \tag{8.-2}$$

where E, F, G are the coefficients of the 1^{st}FF w.r.t. $\boldsymbol{x}$. Here $\boldsymbol{f}_u = \partial_u(f \circ \boldsymbol{x})$ and $\boldsymbol{f}_v = \partial_v(f \circ \boldsymbol{x})$ and $(u, v) \in U$ are the parameter coordinates).

Remark.

(a) If we denote by $\widetilde{E}$, $\widetilde{F}$ and $\widetilde{G}$ the coefficients of the 1^{st}FF of $\widetilde{S}$ w.r.t. the parametrization $\widetilde{\boldsymbol{x}} = f \circ \boldsymbol{x}\colon U \longrightarrow \widetilde{S}$, then we can rephrase this as

$$\widetilde{E} = E, \quad \widetilde{F} = F \quad \text{and} \quad \widetilde{G} = G.$$

(b) A similar result holds for conformal maps: f is conformal on $\boldsymbol{x}(U)$ iff there exists a smooth map $\mu\colon U \longrightarrow (0, \infty)$ such that

$$\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle = \mu E, \quad \langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle = \mu F \quad \text{and} \quad \langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle = \mu G,$$

Example 8.16. (a) Spheres of distinct radii are conformally equivalent (but not isometric, will see this later).

(b) **Gauss map of the catenoid is conformal.** We have seen in Example 8.10 (b) and previous examples that for the parametrization $\boldsymbol{x}$ given by

$$\boldsymbol{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v),$$

the coefficients of the 1$^{\text{st}}$FF are

$$E = G = \cosh^2 v \quad \text{and} \quad F = 0.$$

Moreover, the derivatives of the Gauss map are

$$\boldsymbol{N}_u = \frac{1}{\cosh v}\begin{pmatrix} -\sin u \\ \cos u \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{N}_v = \frac{1}{\cosh^2 v}\begin{pmatrix} -\cos u \sinh v \\ -\sin u \sinh v \\ -1 \end{pmatrix}.$$

Now,

$$\begin{aligned} \langle \boldsymbol{N}_u, \boldsymbol{N}_u \rangle &= \frac{1}{\cosh^2 v} = \frac{1}{\cosh^4 v} E, \quad \langle \boldsymbol{N}_u, \boldsymbol{N}_v \rangle = 0 = F \quad \text{and} \\ \langle \boldsymbol{N}_v, \boldsymbol{N}_v \rangle &= \frac{\sinh^2 v + 1}{\cosh^4 v} = \frac{1}{\cosh^2 v} = \frac{1}{\cosh^4 v} G, \end{aligned}$$

so $\boldsymbol{N}$ is a conformal map with conformal factor (in local parametrization) μ given by $\mu(u,v) = 1/\cosh^4(v)$.