SINGLE MATHS A (MATH 1561)
Matrices terminology - for reference

Here's a quick guide to some of the definitions and a few results for matrices.

Basic definitions:

- An $M \times N$ **matrix** A is an array of M rows and N columns of numbers. The numbers are called **matrix elements** or entries. The element a_{ij} is in the ith row and jth column.
- A **square** matrix has the same number of rows and columns.
- In **matrix multiplication** $P = AB$, we multiply rows of A by columns of B. The product is only defined if A is $M \times N$ and B is $N \times L$; P is then an $M \times L$ matrix. In components,
$$P_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}.$$
- The multiplication is not **commutative**, that is $AB \neq BA$ in general.
- The **identity** is a square matrix I with components $I_{ij} = \delta_{ij}$. It is the identity element for matrix multiplication: $AI = IA = A$ for any matrix A (whenever the product is defined).
- The **transpose** interchanges rows and columns. If A has elements a_{ij}, the transpose A^T has elements $A^T_{ij} = a_{ji}$.
- The **Hermitian conjugate** is the complex conjugate of the transpose. If A has elements a_{ij}, the Hermitian conjugate $A^\dagger$ has elements $A^\dagger_{ij} = a^*_{ji}$.
- A system of linear equations can be written as $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x}$, $\boldsymbol{b}$ are vectors, that is, for our purposes $N \times 1$ matrices. (Row vectors $\boldsymbol{x}^T$ are $1 \times N$ matrices).
- A **homogeneous** system has $\boldsymbol{b} = 0$, so $A\boldsymbol{x} = \mathbf{0}$. The solution of a homogeneous system is said to be in the **kernel** of A. Homogeneous systems always have at least one solution, $\boldsymbol{x} = \mathbf{0}$.

Determinant and inverse:

- The **trace** of a square matrix A is the sum of the diagonal entries: $\mathrm{tr}(A) = \sum_{i=1}^{N} a_{ii}$. The trace of a product of matrices is invariant under cyclic permutations, e.g. $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$.
- The **determinant** of a square matrix A is written $\det(A)$ or $|A|$. It can be defined iteratively:
  - For a 2×2 matrix A, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.
  - The **minor** M_{ij} of a matrix A is the determinant of the $(N-1) \times (N-1)$ matrix formed by removing the ith row and jth column of A.
  - The **cofactor** $C_{ij} = (-1)^{i+j} M_{ij}$.
  - The determinant of A is then obtained by taking the sum over any row or column in A of the product of the elements and their cofactors:
  $$\det(A) = \sum_{j=1}^{N} a_{ij} C_{ij}, \quad \text{for any} \quad i = 1 \ldots N,$$
  or
  $$\det(A) = \sum_{i=1}^{N} a_{ij} C_{ij}, \quad \text{for any} \quad j = 1 \ldots N.$$

- Properties of the determinant:
  - $|A^T| = |A|$; $|A^\dagger| = |A|^*$.
  - $|AB| = |A||B|$.
  - If two rows of A are interchanged to obtain A', $|A'| = -|A|$.
  - Hence if two rows or columns of A are identical, $|A| = 0$.
  - If a row or column of A is added to another row or column, the determinant is unchanged.
  - If a row of A is multiplied by a number λ to obtain A', $|A'| = \lambda|A|$.
  - Hence $|\lambda A| = \lambda^N|A|$, where λ is a number.
- If $|A| = 0$, A is **singular**; if $|A| \neq 0$, A is **non-singular**. For non-singular matrices, the kernel is just $\boldsymbol{x} = \mathbf{0}$.
- The **rank** is the number of rows or columns of a general $M \times N$ matrix which are linearly independent; it is equal to the dimension of the largest square submatrix with a non-zero determinant.
- The **inverse** of a non-singular square matrix A is a matrix A^{-1} such that

  $$AA^{-1} = A^{-1}A = I.$$

  Note that singular matrices and matrices which are not square do not have inverses.
- The inverse is given by $A^{-1} = |A|^{-1}C^T$, where C is the matrix of cofactors of elements of A.

Special matrices:

- A **diagonal** matrix has $a_{ij} = 0$ if $i \neq j$.
- An **upper triangular** matrix has $a_{ij} = 0$ if $i > j$.
- A **lower triangular** matrix has $a_{ij} = 0$ if $i < j$.
- A **symmetric** matrix has $A^T = A$. Similarly an anti-symmetric matrix has $A^T = -A$.
- A **Hermitian** matrix has $A^\dagger = A$. Similarly an anti-Hermitian matrix has $A^\dagger = -A$.
- An **orthogonal** matrix has $A^T = A^{-1}$. Note this implies $\det(A) = \pm 1$.
- A **unitary** matrix has $A^\dagger = A^{-1}$. Note this implies $\det(A)$ has unit modulus.
- A **normal** matrix has $A^\dagger A = AA^\dagger$. Hermitian and unitary matrices are normal.

Eigenvalues and eigenvectors:

- If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, then $\boldsymbol{x}$ is an **eigenvector** of the matrix A with **eigenvalue** λ.
- The eigenvalues λ_i are the roots of the **characteristic equation**: $\det(A - \lambda I) = 0$. This is an Nth order polynomial equation, so in general it will have N complex solutions.
- If the characteristic equation has repeated roots, so the eigenvalues are not all distinct, that is $\lambda_i = \lambda_j$ for some i, j, these eigenvalues are called **degenerate**.
- For non-degenerate eigenvalues, the corresponding eigenvectors are linearly independent.
- A **defective** matrix is one which has less than N linearly independent eigenvectors.
- If A has N linearly independent eigenvectors $\boldsymbol{x}_i$, then $S^{-1}AS$ is diagonal when S is the matrix whose columns are these eigenvectors, that is $S = (\boldsymbol{x}_1 \ldots \boldsymbol{x}_N)$.
- The vectors $\boldsymbol{x}_i$ are **orthogonal** if $\boldsymbol{x}_i^T\boldsymbol{x}_j = \delta_{ij}$.
- If the $\boldsymbol{x}_i$ are orthogonal, then the matrix S formed above is orthogonal.