SINGLE MATHS A (MATH 1561)
Series terminology - for reference

Here's a quick guide to some of the definitions and a few results for series.

Infinite number series

- Infinite series

$$\sum_{k=0}^{\infty} a_k = a_1 + a_2 + ... + a_k + ...$$

  with real or complex entries is called *convergent* if the sequence of its *partial sums*

$$S_n = a_1 + a_2 + ... + a_n = \sum_{k=1}^{n} a_k$$

  has a finite limit $S = \lim_{n\to\infty} S_n$ (otherwise $\sum_{k=0}^{\infty} a_k$ is *divergent*).

- **Necessary condition for convergence:** if a series $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. The converse is not true.

- **Tests for convergence/divergence of series with** *real non-negative* **entries:**
  - **Comparison test:** let $a_k \le b_k$ for all $k > k_0$ for some $k_0 \in \mathbb{N}$.
    * If $\sum b_k$ converges, then $\sum a_k$ converges.
    * If $\sum a_k$ diverges, then $\sum b_k$ diverges.
  - **Quotient test:** let $p = \lim_{k\to\infty} \frac{a_k}{b_k}$.
    * If $p = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
    * If $p = \infty$ and $\sum a_k$ converges, then $\sum b_k$ converges.
    * If p is finite non-zero, then either both $\sum b_k$ and $\sum a_k$ converge, or both diverge.
  - **D'Alembert's ratio test:** let $p = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.
    * If $p < 1$ then $\sum a_k$ is convergent.
    * If $p > 1$ or $p = \infty$ then $\sum a_k$ is divergent.
    * If $p = 1$ or there is no limit, then either may happen.
  - **Cauchy's root test:** let $p = \lim_{k\to\infty} \sqrt[k]{a_k}$.
    * If $p < 1$ then $\sum a_k$ is convergent.
    * If $p > 1$ or $p = \infty$ then $\sum a_k$ is divergent.
    * If $p = 1$ or there is no limit, then either may happen.
  - **Cauchy's integral test:** If $f(x)$ is positive, decreasing and continuous for all $x \ge k_0$ for some $k_0 \in \mathbb{N}$, then $\sum_{k=k_0}^{\infty} f(k)$ converges if and only if $\int_{k_0}^{\infty} f(x)\,dx < \infty$

- **Remark:** there is **no algorithm** telling you which test to apply for a given series. The only way to get the intuition how to find a right one is through doing a lot of exercises on the convergence of series.

- Main example series to compare with:

  $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$;

  $\sum_{k=1}^{\infty} \frac{1}{k^r}$ converges iff $r > 1$;

  $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ converges iff $|q| < 1$ (holds for negative q as well).

- A series $\sum a_k$ with real or complex entries is called
  - **absolutely** convergent if $\sum |a_k|$ converges.
  - **conditionally** convergent if $\sum |a_k|$ diverges, but $\sum a_k$ converges.
- **Fact:** absolutely convergent series converges.
- A series $\sum a_k$ with real entries is *alternating* if for every k the signs of a_k and a_{k+1} are distinct.
- **Leibniz's test for convergence of alternating series:** let

  $$\sum a_k = \sum (-1)^k b_k = -b_1 + b_2 - b_3 + b_4 - ... \qquad (b_k \geq 0),$$

  and assume that

  $$\lim_{k\to\infty} b_k = 0 \qquad \text{and} \qquad b_k \geq b_{k+1} \quad \text{for every } k$$

  Then $\sum a_k$ converges.
- **Operations with series**
  - A convergent series $S = \sum a_k$ may be multiplied by any number α: $\sum \alpha a_k = \alpha S$.
  - For two convergent series $S = \sum a_k$ and $T = \sum b_k$ a sum of them is also a convergent series $\sum (a_k + b_k) = S + T$.
  - If a series converges absolutely, any rearrangement of the terms leads to a convergent series with the same sum. This is not true for conditionally convergent series.

Power series

- *Power series* is a series of the form

  $$P(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + ... + a_k(x-c)^k + ... = \sum_{k=0}^{\infty} a_k (x-c)^k,$$

  where a_i and c are real or complex numbers. If $c = 0$, the sum is of the form $P(x) = \sum_{k=0}^{\infty} a_k x^k$.
- **Convergence of power series**
  For given $x = x_0$ a power series $P(x_0)$ is a number series, so it is either convergent or divergent. For any power series there exists a *disc of convergence* $|x - c| < R$ (an *interval of convergence* $(c - R, c + R)$ in case of real a_i and c), such that $P(x)$ is convergent for any x from this interval. The value R is called *radius of convergence*, and can be found using quotient or root test:

  $$R = \frac{1}{\lim\limits_{k\to\infty} \sqrt[k]{|a_k|}} \qquad \text{and} \qquad R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

  A power series converges absolutely inside its disc (or interval) of convergence. Convergence at the boundary of the disc should be considered separately.

- **Operations with power series**
  Inside the disc (or interval) of convergence, power series can be differentiated and integrated.

Taylor series

- **Taylor polynomials**
  Let $f(x)$ be an infinitely differentiable function at the neighborhood of a point $c \in \mathbb{R}$. A *Taylor polynomial* of $f(x)$ of degree n centered at c is a polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + ... + f^{(n)}(c)\frac{(x-c)^n}{n!} = \sum_{k=0}^{n} f^{(k)}(c)\frac{(x-c)^k}{k!}$$

- **Taylor series**
  Taylor series of $f(x)$ is a power series

$$P(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + ... + f^{(n)}(c)\frac{(x-c)^n}{n!} + ... = \sum_{k=0}^{\infty} f^{(k)}(c)\frac{(x-c)^k}{k!}$$

  If the function $f(x)$ is good enough, $P(x)$ converges to $f(x)$ for x in the interval of convergence.

- **Lagrange form of the remainder**
  The *remainder* R_n is defined by

$$R_n(x) = f(x) - P_n(x)$$

  Taylor's theorem gives an explicit form (*Lagrange form*) of $R_n(x)$:

$$R_n(x) = f^{(n+1)}(\xi)\frac{(x-c)^{n+1}}{(n+1)!}$$

  for some $\xi \in (c, x)$ (or $\xi \in (x, c)$). This form (usually) allows to approximate a function by its Taylor polynomials.