Lecture 10

Taylor series and limits

Taylor series can be used to calculate limits of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)},$$

where both $f(x)$ and $g(x)$ go to 0 as $x \to a$. Sometimes, but not always, l'Hôpital's rule can also be used.

Example 10.1. Compute $\lim_{x\to 0}\frac{\sin x}{x}$. First consider $\sin x$ for x near 0. By (8.3), we can write

$$\sin x = x - \frac{x^3}{3!} + o(x^3),$$

where $o(x^3)$ denotes terms which are of order x^4 and higher (because higher powers like x^4 and further go to 0 faster than x^3, i.e. $\lim_{x\to 0}\frac{o(x^3)}{x^3} = 0$). Then we can write

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x - \frac{x^3}{3} + o(x^3)}{x} = \lim_{x\to 0}\frac{x(1 - \frac{x^2}{3} + \frac{o(x^3)}{x})}{x} = \lim_{x\to 0} 1 - \frac{x^2}{3} + \frac{o(x^3)}{x} = 1,$$

since

$$\lim_{x\to 0}\frac{o(x^3)}{x} = \lim_{x\to 0}\frac{o(x^3)}{x^3}x^2 = \lim_{x\to 0}\frac{o(x^3)}{x^3}\lim_{x\to 0}x^2 = 0\cdot 0 = 0.$$

Remark. Note that the $o(x^n)$ does not denote a concrete function, but any (converging) power series $\sum a_k x^k$ around 0 such that the coefficients $a_0, \dots a_n$ are all equal to zero (as this guarantees that $\lim_{x\to 0}\frac{\sum a_k x^k}{x^n} = 0$). For example, $o(x^5)$ is simultaneously $o(x^3)$ as $\lim_{x\to 0}\frac{o(x^5)}{x^5} = 0$ implies

$$\lim_{x\to 0}\frac{o(x^5)}{x^3} = \lim_{x\to 0}\frac{o(x^5)}{x^5}x^2 = \lim_{x\to 0}\frac{o(x^5)}{x^5}\lim_{x\to 0}x^2 = 0\cdot 0 = 0.$$

(but the converse may not be true of course!), $\frac{o(x^n)}{x} = o(x^{n-1})$ (as we can see in the example above for $n = 3$), $o(x^n)\cdot x^m = o(x^{n+m})$ for general m, n, and $o(x^n) + o(x^n) = o(x^n)$.

Here is an example where we cannot use l'Hôpital's rule, and where Taylor series works (see Q22 from the problem sheet)

Example 10.2. Let

$$f(x) = \exp\left(\frac{\sin x}{1-3x}\right)$$

and calculate

$$\lim_{x\to 0}\frac{f(x) - (x+1)}{x\cos x - \ln(1+x)}. \tag{limit}$$

Trying l'Hôpital leads to a horrible mess which is not easier than the original function. Use Taylor series instead.

First consider $f(x)$ for x near 0 ($|x| < 1$). By (8.3) and the geometric series $\frac{1}{1-x} = 1+x+x^2+\cdots$ for $|x| < 1$, we can write

$$\sin x = x - \frac{x^3}{3!} + o(x^3), \qquad \text{and} \qquad \frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + o(x^3),$$

Thus

$$\begin{aligned}\frac{\sin x}{1-3x} &= \left(x - \frac{x^3}{3!} + o(x^3)\right)(1 + 3x + 9x^2 + 27x^3 + o(x^3)) \\ &= x - \frac{x^3}{3!} + o(x^3) + 3x^2 + 9x^3 + o(x^3) \\ &= x + 3x^2 + \frac{53}{6}x^3 + o(x^3).\end{aligned}$$

We also know that $e^x = 1 + x + \frac{x^2}{2} + \cdots$ and plugging in the above, we get

$$\begin{aligned}f(x) = 1 + x + 3x^2 + \frac{53}{6}x^3 + o(x^3) &+ \left(x + 3x^2 + \frac{53}{6}x^3 + o(x^3)\right)^2/2 \\ &+ \left(x + 3x^2 + \frac{53}{6}x^3 + o(x^3)\right)^3/6 + o(x^3) \\ &= 1 + x + 3x^2 + \frac{53}{6}x^3 + o(x^3) + x^2/2 + (2x \cdot 3x^2)/2 + x^3/6 + o(x^3) \\ &= 1 + x + \frac{7}{2}x^2 + 12x^3 + o(x^3).\end{aligned}$$

To calculate the final limit, we also need to Taylor expand $\cos x\ \ln(1+x)$ up to order 3:

$$\cos x = 1 - \frac{x^2}{2} + o(x^3), \qquad \text{and} \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3),$$

so $x\cos x - \ln(1+x) = \frac{x^2}{2} - \frac{5}{6}x^3 + o(x^3)$. Thus,

$$(\text{limit}) = \lim_{x\to 0} \frac{\frac{7}{2}x^2 + 12x^3 + o(x^3)}{\frac{x^2}{2} - \frac{5}{6}x^3 + o(x^3)} = \frac{\frac{7}{2} + 12x + o(x)}{\frac{1}{2} - \frac{5}{6}x + o(x)} = \frac{7/2}{1/2} = 7.$$

Another example of computing limits.

Example 10.3.

$$\lim_{x\to 0} \frac{\frac{x^2}{1-x} + 2\cos x - 2}{2x^3 + 3x^7} = \lim_{x\to 0} \frac{x^2(\sum x^k) + 2\cos x - 2}{2x^3 + 3x^7} =$$
$$\lim_{x\to 0} \frac{x^2 + x^3 + o(x^3) + 2(1 - \frac{x^2}{2} + o(x^3)) - 2}{2x^3 + 3x^7} = \lim_{x\to 0} \frac{x^2 + x^3 + o(x^3) - x^2 + o(x^3)}{2x^3 + o(x^3)} =$$
$$\lim_{x\to 0} \frac{x^3 + o(x^3)}{2x^3 + o(x^3)} = \lim_{x\to 0} \frac{1 + \frac{o(x^3)}{x^3}}{2 + \frac{o(x^3)}{x^3}} = \frac{\lim_{x\to 0} 1 + \frac{o(x^3)}{x^3}}{\lim_{x\to 0} 2 + \frac{o(x^3)}{x^3}} = \frac{1 + \lim_{x\to 0} \frac{o(x^3)}{x^3}}{2 + \lim_{x\to 0} \frac{o(x^3)}{x^3}} = \frac{1+0}{2+0} = 1/2.$$