Lecture 13

Example 13.1. Here is another situation with three variables and three equations.

$$\left\{\begin{array}{rcl} x+y &=& 3 \\ y+z &=& 1 \\ x-z &=& 2 \end{array}\right. \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 \end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right)$$

$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This is equivalent to the system

$$\left\{\begin{array}{rcl} x-z &=& 2 \\ y+z &=& 1 \end{array}\right. \iff \left\{\begin{array}{rcl} x &=& 2+z \\ y &=& 1-z \end{array}\right. .$$

For any $z \in \mathbb{R}$ (without restriction) we thus have a solution

$$(x, y, z) = (2+z, 1-z, z).$$

Hence, there are *infinitely many* solutions to this system.

Example 13.2.

$$\left\{\begin{array}{rcl} 2x+2y+3z &=& 7 \\ x+2y-z &=& 0 \end{array}\right. \longrightarrow \left(\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 1 & 2 & -1 & 0 \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|c} 0 & -2 & 5 & 7 \\ 1 & 2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -2 & 5 & 7 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 7 \\ 0 & -2 & 5 & 7 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 7 \\ 0 & 1 & -5/2 & -7/2 \end{array}\right) \longrightarrow \left\{\begin{array}{rcl} x+4z &=& 7 \\ y-5z/2 &=& -7/2 \end{array}\right. .$$

$$\iff \left\{\begin{array}{rcl} x &=& 7-4z \\ y &=& -7/2+5z/2 \end{array}\right. .$$

So for each $z \in \mathbb{R}$ we have a solution for x and y. Thus the system has infinitely many solutions.

Gauss elimination

This is the general algorithm to solve a linear system of equations. We have already seen the method in several examples, but we will describe exactly what its goal is, and give more examples.

In order to solve a system $(A \mid \underline{b})$, we use ERO to transform this matrix into one where A is *as close as possible* to an identity matrix. Looking back at some of the previous examples, we got:

- Example (12.5):
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right).$$
(unique solution)
- Example (13.1):
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$
(infinitely many solutions)

It is not always possible to achieve an identity matrix, but something close to it is always possible, namely a matrix such that:

- Every row starts with some 0-entries (possibly no 0-entries for the first row) followed by a 1-entry (unless the whole row is zero).
- The number of 0-entries in a row is *more* than in the preceding row (unless both the row and the preceding one consist of 0-entries only).
- The first non-zero entry in each row lies below 0-entries of all previous rows.

Definition 13.3. Augmented matrices satisfying the above three conditions are said to be in *Reduced Row Echelon Form (RREF).*

In (13.4), the matrix
$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
is in RREF. We have no zeros in the beginning of the 1st row, two zeros in the 2nd, and three in the 3rd. Note that the first non-zero entry in the 2nd row lies below a 0-entry of the 1st row.

Here is an example to illustrate RREF:

Example 13.4.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 3 & -1 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 2 & 3 & -1 \end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right)$$
$$\xrightarrow{-R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{array}\right) \xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\iff \left\{ \begin{array}{rcl} x & = & 2-2y \\ z & = & -1. \end{array}\right.$$

Thus, we have have infinitely many solutions (one for each $y \in \mathbb{R}$).

Here is another example, which shows how to apply the ERO systematically in Gaussian elimination to achieve RREF.

Example 13.5.

$$\left\{\begin{array}{rcl} x+2y+z &=& 3 \\ 2x+y-z &=& -3 \\ x+y+2z &=& 4 \\ 2x+3y+3z &=& 7 \end{array}\right. \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & -3 \\ 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 7 \end{array}\right) \xrightarrow{\substack{R_2-2R_1 \\ R_3-R_1 \\ R_4-2R_1}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & -3 & -9 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

(we did this to clear the entries below the first 1-entry)

$$\xrightarrow{-\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{\substack{R_3+R_2 \\ R_4+R_2}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{array}\right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_4-2R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\substack{R_1-R_3 \\ R_2-R_3}} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This last matrix is in RREF, and the matrix to the left of the vertical line is an identity matrix with an extra zero-row added. This means that the system has a *unique* solution:

$$x=-1, \quad y=1, \quad z=2.$$