Lecture 14

Structure of solutions of linear systems

The discussion and examples so far show that three cases can appear:

- A unique solution: The RREF looks like an identity matrix with extra zero-rows:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{array}\right).$$

- No solutions: The RREF has a row with only zeros on the left and a non-zero entry on the right.

- Infinitely many solutions: The number of rows in the RREF is *smaller* than the number of columns (when zero rows are ignored).

A linear system

$$A\underline{x} = \underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

is called *homogeneous*. The set of solutions is then called the *kernel* of the matrix A. There is always at least one solutions: $\underline{x} = \underline{0}$. If the RREF of $(A \mid \underline{0})$ has less rows than columns, then the system $A\underline{x} = \underline{0}$ also has non-zero solutions.

Now, suppose we have a linear system

$$A\underline{x} = \underline{b},$$

and let $\underline{x}_1$ and $\underline{x}_2$ be two solutions, that is, $A\underline{x}_1 = \underline{b}$ and $A\underline{x}_2 = \underline{b}$. Then

$$A(\underline{x}_1 - \underline{x}_2) = \underline{b} - \underline{b} = \underline{0}.$$

Thus the difference of any two solutions is a solutions to a homogeneous system. In particular, if we fix one solution $\underline{x}_0$ such that $A\underline{x}_0 = \underline{b}$, then *any* solution of $A\underline{x} = \underline{b}$ has the form

$$\underline{x} = \underline{x}_0 + \underline{x}_h,$$

where $\underline{x}_h$ runs through all the solutions to the homogeneous system $A\underline{x} = \underline{0}$.

Thus, to solve the system $A\underline{x} = \underline{b}$, we need to know:

- *one* solution to $A\underline{x} = \underline{b}$,

- all the solutions to $A\underline{x} = \underline{0}$.

Example 14.1.

$$\left\{\begin{array}{rcl} x+y &=& 1 \\ 2x+2y &=& 2 \end{array}\right. \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \Longleftrightarrow x+y=1.$$

So one solution is, for example, $\underline{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now, the solutions of $A\underline{x} = \underline{0}$ are given by

$$x + y = 0,$$

that is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Thus, the solutions of the original system are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ 1-\lambda \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

Systems with a parameter

Example 14.2. Find the values of $\lambda \in \mathbb{R}$ such that the following system has no solutions, one solution, or infinitely many solutions:

$$\left\{\begin{array}{ccc} x+\lambda y+z &=& 1 \\ \lambda x+y+(\lambda-1)z &=& \lambda \\ x+y+z &=& \lambda+1 \end{array}\right. .$$

We use ERO to put matrix of the system into RREF:

$$\left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ \lambda & 1 & \lambda-1 & \lambda \\ 1 & 1 & 1 & \lambda+1 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-\lambda R_1} \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1-\lambda^2 & -1 & 0 \\ 0 & 1-\lambda & 0 & \lambda \end{array}\right) \xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1-\lambda & 0 & \lambda \\ 0 & 1-\lambda^2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3-(1+\lambda)R_2} \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1-\lambda & 0 & \lambda \\ 0 & 0 & -1 & -\lambda(1+\lambda) \end{array}\right) \xrightarrow[-R_3]{R_1+R_3} \left(\begin{array}{ccc|c} 1 & \lambda & 0 & 1-\lambda(1+\lambda) \\ 0 & 1-\lambda & 0 & \lambda \\ 0 & 0 & 1 & \lambda(1+\lambda) \end{array}\right).$$

To go further, we would have to divide the second row by $1-\lambda$. This can only be done if $1-\lambda \neq 0$, that is, if $\lambda \neq 1$, so let's assume this for the moment. If $\lambda \neq 1$:

$$\xrightarrow{\frac{1}{1-\lambda}R_2} \left(\begin{array}{ccc|c} 1 & \lambda & 0 & 1-\lambda(1+\lambda) \\ 0 & 1 & 0 & \frac{\lambda}{1-\lambda} \\ 0 & 0 & 1 & \lambda(1+\lambda) \end{array}\right) \xrightarrow{R_1-\lambda R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1-\lambda(1+\lambda)-\frac{\lambda^2}{1-\lambda} \\ 0 & 1 & 0 & \frac{\lambda}{1-\lambda} \\ 0 & 0 & 1 & \lambda(1+\lambda) \end{array}\right).$$

Since the left side in the RREF is an identity matrix, the system has one solution in this case.

We now need to consider the case $\lambda = 1$: After the second step above, we then have the matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

This is an inconsistent system, which has no solutions.

Conclusion: For $\lambda \neq 1$ there is a unique solution, namely

$$\begin{aligned} x &= 1 - \lambda(1+\lambda) - \frac{\lambda^2}{1-\lambda} \\ y &= \frac{\lambda}{1-\lambda} \\ z &= \lambda(1+\lambda). \end{aligned}$$

For $\lambda = 1$ there are no solutions, and for no value of λ does the system have infinitely many solutions.

Note: in the second step above, we note that $(1-\lambda^2) = (1-\lambda)(1+\lambda)$. Instead of steps 2 and 3, we could have divided the row

$$\left(\begin{array}{ccc|c} 0 & 1-\lambda & 0 & \lambda \end{array}\right)$$

by $1-\lambda$ in order to get a 1 after the first 0. However, then we would have to assume that $\lambda \neq 1$, because we can't divide by 0. We would therefore have to consider two cases: first $\lambda \neq 1$ and then $\lambda = 1$. This is a perfectly fine approach, but here we tried to avoid splitting into cases for as long as we could.

Example 14.3. Find the values of $\lambda \in \mathbb{R}$ such that the following system has no solutions, one solution, or infinitely many solutions:

$$\left\{\begin{array}{rcl} x+y+z &=& -1 \\ \lambda x + y + z &=& -1 \\ x + \lambda^2 y + z &=& \lambda \end{array}\right. .$$

Again, we use ERO to simplify the augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ \lambda & 1 & 1 & -1 \\ 1 & \lambda^2 & 1 & \lambda \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - \lambda R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1-\lambda & 1-\lambda & \lambda - 1 \\ 0 & \lambda^2 - 1 & 0 & \lambda + 1 \end{array}\right)$$

If $\lambda = 1$, this results in $\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$, which has no solution.

Otherwise, we continue:

$$\xrightarrow{R_2/(1-\lambda)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & \lambda^2 - 1 & 0 & \lambda + 1 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & \lambda^2 - 1 & 0 & \lambda + 1 \end{array}\right)$$

Now, if $\lambda = -1$, this results in $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ which has infinitely many solutions.

Otherwise, we continue:

$$\xrightarrow{R_3/(\lambda+1)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & \lambda - 1 & 0 & 1 \end{array}\right) \xrightarrow[R_3/(\lambda-1)]{R_2 - R_3/(\lambda-1)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 - 1/(\lambda - 1) \\ 0 & 1 & 0 & 1/(\lambda - 1) \end{array}\right)$$

which has a unique solutions.

Summarizing, the system has no solutions if $\lambda = 1$, infinitely many solutions if $\lambda = -1$, and a unique solution otherwise.