Lecture 15

The determinant

The determinant is an important number associated to a square matrix denoted by $\det A$ or $|A|$. We will define the determinant recursively. First, for a 1×1 matrix $A = (a)$ we set the determinant $\det A = a$. For an $n \times n$ matrix, we will now define the determinant in terms of determinants of $(n-1) \times (n-1)$ matrices.

i) Let $A = (a_{ij})$, for $1 \leq i, j \leq n$.

ii) For each i and j, let M_{ij} be the determinant of the matrix obtained by removing the ith row and the jth column of A.

iii) The *determinant* of A, denoted $\det A$ or $|A|$, is

$$\begin{aligned}\det A &= a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} + \cdots + (-1)^{n+1}a_{n1}M_{n1} \\ &= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} M_{i1}.\end{aligned}$$

The numbers $C_{ij} = (-1)^{i+j}M_{ij}$ are called *cofactors* of matrix A. The definition can be written in the form

$$\det A = \sum_{i=1}^{n} a_{i1}C_{i1},$$

this expression is called the *expansion of* $\det A$ *along the first column.*

Example 15.1. Compute $\det A$, where $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$. We have

$$M_{11} = \det(5) = 5, \qquad M_{21} = \det(2) = 2.$$

Thus $\det A = 1 \cdot 5 - 2 \cdot 2 = 1$.
In general,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Fact. We can compute $\det A$ using expansion along *any* column: given $j = 1, \ldots, n$, we have

$$\det A = \sum_{i=1}^{n} a_{ij}C_{ij},$$

Moreover, we can also use *expansions along rows*: given $i = 1, \ldots, n$, we have

$$\det A = \sum_{j=1}^{n} a_{ij}C_{ij},$$

Example 15.2. Compute $\det A$, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$. Using expansion along the third row, we get

$$\det A = 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -2(-2) + 3 = 7.$$

If we expand along the second column, we obtain

$$\det A = -0 \cdot \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 3 - 2(-2) = 7.$$

Properties of the determinant

0) Determinant does not depend on the choice of the row or column in the expansion.

1) $\det I_n = 1$.

2) $\det A^T = \det A$.

3) Multiplicative: if A and B are two square matrices, we have

$$\det(AB) = \det A \cdot \det B.$$

4) If $A \in \mathrm{Mat}_n$ is diagonal, then $\det A = \prod_{i=1}^n a_{ii}$.

5) A matrix A is called *upper-triangular* if $a_{ij} = 0$ for all $i > j$. For upper-triangular matrices $\det A = \prod_{i=1}^n a_{ii}$ as well.

Note that we know how to compute the determinant of an upper-triangular matrix, and we also know how to transform a matrix to an upper-triangular form by row operations. We are now interested in the following question: what is the behavior of the determinant under ERO?