# Lecture 16

### Matrices of ERO

ERO can be written in terms of multiplication of matrices. Denote by $I_{ij} \in \mathrm{Mat}_n$ the $n \times n$ matrix with $(ij)$-element equal to 1 and all other elements being zero. Let $A \in \mathrm{Mat}_n$. Then:

- ERO of type 1, i.e. adding row $k$ of $A$ multiplied by $\lambda$ to row $i$ of $A$, is equivalent to multiplying $A$ by $I + \lambda I_{ik}$ from the left:
$$A \xrightarrow{R_i + \lambda R_k} (I + \lambda I_{ik})A$$
Note that $\det(I + \lambda I_{ik}) = 1$, so by the multiplicative property of the determinant EROs of first type leave the determinant intact.

- ERO of type 2, i.e. swapping rows $i$ and $k$ of $A$, is equivalent to multiplying $A$ by $I - I_{ii} - I_{kk} + I_{ik} + I_{ki}$ from the left:
$$A \xrightarrow{R_i \leftrightarrow R_k} (I - I_{ii} - I_{kk} + I_{ik} + I_{ki})A$$
Since $\det(I - I_{ii} - I_{kk} + I_{ik} + I_{ki}) = -1$, EROs of second type change the sign of the determinant.

- ERO of type 3, i.e. multiplying row $i$ of $A$ by $\lambda$, is equivalent to multiplying $A$ by $I + (\lambda - 1)I_{ii}$ from the left:
$$A \xrightarrow{\lambda R_i} (I + (\lambda - 1)I_{ii})A$$
Since $\det(I + (\lambda - 1)I_{ii}) = \lambda$, EROs of third type multiply the determinant by $\lambda$.

Therefore, we can compute the determinant by Gauss elimination. Note that we can always transform a matrix to an upper-triangular form without using ERO of type 3 (though sometimes we may use them for convenience).

**Example 16.1.** Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -4 & -5 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{vmatrix} = -(-5) = 5.$$

**Example 16.2.** Let $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & 5 \\ -2 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{pmatrix}$. Then

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & 5 \\ -2 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 2 \\ 0 & 4 & 5 & 4 \\ 0 & 1 & -2 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 4 & 5 & 4 \\ 0 & -5 & -1 & 2 \end{vmatrix} =$$

$$- \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & -11 & 7 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 7 \end{vmatrix} = -13 \cdot 7 = -91$$

**Algorithm**

Use Gauss elimination to transform a matrix $A$ to an upper-triangular matrix $T$ by ERO of types 1 and 2. Then $\det A = (-1)^m \det T$, where $m$ is the number of ERO of type 2 we have applied.

**Further properties of determinant**

6) If $A$ contains a zero row (or column) then $\det A = 0$.

7) If $A$ contains two similar rows (or columns) then $\det A = 0$.

## The inverse of a matrix

Let $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{R})$ be an $n \times n$ matrix, and $I = I_n$ the identity matrix (of size $n$). Then

$$AI = IA = A.$$

In other words, the identity matrix does not change another matrix, when multiplied.

Now, let $A\underline{x} = \underline{b}$ be a linear system and suppose that there is a matrix $B$ such that

$$BA = I.$$

Then we can multiply by $B$ on both sides:

$$BA\underline{x} = B\underline{b} \Longleftrightarrow \underline{x} = B\underline{b}.$$

This says that there exists a unique solution $\underline{x} = B\underline{b}$ to the system.

**Definition 16.3.** Let $A$ be a square matrix. A matrix $B$ is called the *inverse* of $A$ if

$$AB = BA = I.$$

The inverse of $A$ (if it exists) is denoted $A^{-1}$.