Lecture 17

Example 17.1. The system

$$\begin{cases} x + 2y &= 1 \\ 2x + 5y &= -3 \end{cases}.$$

can be written $A\underline{x} = \underline{b}$, where $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Let

$$B = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}.$$

We can check that $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. Thus the solution is

$$\underline{x} = B\underline{b} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \end{pmatrix}.$$

So, we see that B is the key to solving the system. Let's now compute the RREF:

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 5 & -3 \end{array} \right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -5 \end{array} \right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & -5 \end{array} \right) = (I \mid B\underline{b}).$$

In particular, the RREF is an identity matrix followed by a solution vector.

We saw above that the inverse of $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ is $\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$. The inverse does not always exist. For example, there is no inverse to the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

A matrix A having an inverse is called *invertible.*

Observe that the matrix above has zero determinant. Determinant and inverse are related by the following statement:

Theorem 17.2. *A matrix A has an inverse if and only if* $\det A \neq 0$.

Why is this theorem true? Assume first that A has an inverse A^{-1}. Then, by the multiplicative property of the determinant,

$$\det A \det A^{-1} = \det(A \cdot A^{-1}) = \det I = 1,$$

which implies that $\det A \neq 0$.

Now we need to understand why does $\det A \neq 0$ imply the existence of the inverse. First, we note the following.

Proposition 17.3. *If* $\det A \neq 0$ *then a system of linear equations $A\underline{x} = \underline{b}$ has a unique solution. In particular, RREF of A is the identity matrix.*

Indeed, observe that RREF of A is obtained from A by ERO, which implies that the determinant of the RREF is a non-zero multiple of $\det A$. Thus, if $\det A \neq 0$, the determinant of RREF is not zero either, so RREF of A does not contain zero rows. Since it is a square matrix, by the definition of RREF we conclude that it is the identity matrix I, and the system has a unique solution.

Now, for every matrix A with RREF of A being the identity matrix we will explicitly construct the inverse.

Algorithm: finding the inverse using ERO

Let $\det A \neq 0$, so that the RREF of A is the identity matrix I_n. Then we can transform A to I_n by ERO, let k be the number of ERO required. Denote by $E_1, \dots, E_k$ the matrices of these ERO. Then

$$E_k E_{k-1} \dots E_1 \cdot A = I_n.$$

Denote $B = E_k E_{k-1} \dots E_1$, then we see that $BA = I_n$, so $B = A^{-1}$ is the required inverse of A! Further, if we create an augmented matrix $(A \,|\, I_n)$ and apply to it the row operations above, then we obtain

$$(A \,|\, I_n) \to (E_k E_{k-1} \dots E_1 \cdot A \,|\, E_k E_{k-1} \dots E_1 \cdot I_n) = (I_n \,|\, A^{-1} I_n) = (I_n \,|\, A^{-1}),$$

which leads to the following

Algorithm. Create the augmented matrix $(A \,|\, I_n)$, apply to it ERO to transform A to I_n. Then the resulting matrix on the right is A^{-1}.

We can summarize the discussion above:

Corollary 17.4. *For $A \in \mathrm{Mat}_n$ the following are equivalent:*

- $\det A \neq 0$*;*
- *RREF of A is the identity matrix I_n;*
- *A is invertible.*

Computing the inverse

Suppose that A is a matrix which has an inverse (i.e., $\det A \neq 0$). To compute the inverse, we use the algorithm above based on the Gauss elimination on an augmented matrix.

Example 17.5. Compute the inverse of $A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$. We perform Gauss elimination on the augmented matrix

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1/3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2-2R_1}$$

$$\xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 1/3 & -2/3 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 1/3 & -2/3 & 1 & 0 \\ 0 & 0 & 1/3 & 4/3 & -2 & 1 \end{array}\right) \xrightarrow[R_2-R_3]{R_1-R_3}$$

$$\xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1/3 & 4/3 & -2 & 1 \end{array}\right) \xrightarrow{3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 4 & -6 & 3 \end{array}\right).$$

Once we reach an identity matrix to the left, we stop. The inverse of A is on the right of the vertical line.

We now verify the answer:

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ 4 & -6 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 17.6. Compute the inverse of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. As in the previous example, we perform Gauss elimination on the augmented matrix

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_2}$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right).$$

The matrix on the left has a zero row, so it cannot be transformed to the identity matrix. Therefore, A is not invertible.

Properties of the inverse

- $(A^{-1})^{-1} = A$ (since $A \cdot A^{-1} = I_n$).
- $(A^T)^{-1} = (A^{-1})^T$ (since $A^T \cdot (A^{-1})^T = (A \cdot A^{-1})^T = I_n$).
- $(AB)^{-1} = B^{-1}A^{-1}$ (since $B^{-1}A^{-1}AB = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$).
- If A^{-1} exists then it is unique.

Cofactor method

Recall that a *cofactor* C_{ij} of a matrix $A \in \mathrm{Mat}_n$ is defined by $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the determinant of an $(n-1) \times (n-1)$-matrix obtained from A by removing i-th row and j-th column. Denote by C the matrix composed of cofactors of A, and consider its transpose C^T, i.e. $(C^T)_{ij} = C_{ij}$.

Example 17.7. If $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, then $C = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$, so $C^T = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$. Thus,

$$AC^T = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_2 = (\det A)\, I_2.$$

In particular, this implies that $A \cdot \frac{1}{\det A} C^T = I_2$, and thus $A^{-1} = \frac{1}{\det A} C^T = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$.