Lecture 19

Unitary and normal matrices A complex matrix A is called **unitary** if $A^\dagger A = I$, i.e. $A^\dagger = A^{-1}$. Note that real orthogonal matrices are also unitary. The set of all unitary $(n\times n)$ matrices is denoted by $U(n)$.

Example 19.1. A matrix $A = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$ is unitary: $A^\dagger = A^* = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$, so $A^\dagger A = I$.

Similarly to orthogonal matrices, an inverse of a unitary matrix is also unitary, and a product of unitary matrices is a unitary matrix. Also,

$$1 = \det I = \det(A^{-1}A) = \det(A^\dagger A) = \det A^\dagger \det A = (\det A)^* \det A = |\det A|^2,$$

so $|\det A| = 1$.

Unitary matrices preserve the length $l(\underline{v})$ of a complex vector $\underline{v} = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix}$ defined by $l(\underline{v}) = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$.

A matrix A is called **normal** if $A^\dagger A = AA^\dagger$, i.e. if it commutes with its Hermitian conjugate. For example, Hermitian and unitary matrices are normal. An inverse of a normal matrix (if exists) is also normal.

Vector spaces

Let

$$\mathbb{R}^n = \{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_i \in \mathbb{R}\}$$

be the set of $n \times 1$ column vectors of real numbers. Similarly, if we replace $\mathbb{R}$ by $\mathbb{C}$ we get $\mathbb{C}^n$. We can add two vectors

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

and we can multiply a vector by a scalar $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$ if we work over $\mathbb{C}$):

$$\lambda \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$

(but note that we cannot multiply two vectors because the matrix product is not defined for two $n \times 1$ matrices, unless $n = 1$!)

Definition 19.2. A vector space is a set with two operations: addition and scalar multiplication. Its elements are called *vectors*. In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces.

Definition 19.3. Let $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m$ be vectors and $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ (or $\mathbb{C}$). The vector

$$\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 + \cdots + \lambda_m \underline{v}_m$$

is called a *linear combination* of $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m$.

Example 19.4. Let $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$. Then any vector $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ is a linear combination of $\underline{v}_1$ and $\underline{v}_2$ because

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x\underline{v}_1 + y\underline{v}_2.$$

Note that the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is *not* a linear combination of $\underline{v}_1$ and $\underline{v}_2$.

A set of vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m$ is called *linearly dependent* if one of the vectors is a linear combination of the others, that is, if

$$\underline{v}_i = \lambda_1 \underline{v}_1 + \cdots + \lambda_{i-1} \underline{v}_{i-1} + \lambda_{i+1} \underline{v}_{i+1} + \cdots + \lambda_m \underline{v}_m,$$

for some $1 \le i \le m$. This is equivalent to saying that there exist scalars $\lambda_1, \dots, \lambda_m$ (not all of them zero!) such that

$$\lambda_1 \underline{v}_1 + \cdots + \lambda_m \underline{v}_m = \underline{0}.$$

If $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m$ are not linearly dependent, they are said to be *linearly independent*. Mathematically, this means that the relation

$$\lambda_1 \underline{v}_1 + \cdots + \lambda_m \underline{v}_m = \underline{0}$$

can only hold if $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$.

Example 19.5.

- $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$ are linearly dependent, because

  $$2\underline{v}_1 - \underline{v}_2 = \underline{0}.$$

- $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent, because if

  $$\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ -\lambda_2 \end{pmatrix} = \underline{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  then we must have

  $$\lambda_1 + \lambda_2 = 0, \quad \text{and} \quad -\lambda_2 = 0,$$

  that is, $\lambda_1 = \lambda_2 = 0$.

- $\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$ are linearly dependent because we have the relation

$$\underline{v}_1 + \underline{v}_2 = \underline{v}_3.$$

Definition 19.6. The *span* of the vectors $\underline{v}_1, \dots, \underline{v}_m$, written

$$\text{span}\{\underline{v}_1, \dots, \underline{v}_m\},$$

is the set of all vectors which are linear combinations of $\underline{v}_1, \dots, \underline{v}_m$.

Example 19.7.

- $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = \mathbb{R}^2$, since for any $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ there are λ_1, λ_2 such that $\begin{pmatrix} a \\ b \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, namely,$\lambda_2 = -b, \quad \lambda_1 = a + b$.
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$$\begin{aligned} \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} &= \left\{\lambda_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \\ &= \left\{(\lambda_1 + \lambda_3) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (\lambda_2 + \lambda_3) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}. \end{aligned}$$