Single Maths A – Epiphany 2018

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All the times and room numbers are the same as last term.

Homeworks will be set on Fridays, during the lecture. Hand in your homeworks in the lockers in CM117. The deadline is strictly 5pm, before the start of the lecture the following week. Tutorials start in week 12.

All the course material will be made available on DUO and on the class page `http://www.maths.dur.ac.uk/users/pavel.tumarkin/SMA`.

WARNING! The lectures will *approximately* correspond to the notes below. However things may change and there will be more material at the end hopefully. The content of the course is defined by the actual lectures, not by the following notes!

Short outline of this term's content

- Series, Taylor series
- Matrices, systems of linear equations
- Vector spaces, linear maps, eigenvalues, eigenvectors
- Groups

Textbook: Same as last term (Riley et al).

Series

Here is a problem about series which is part of a popular story[1]:

> Two trains are 20 miles apart on the same track heading towards each other at 10 mi/h, on a collision course. At the same time, a fly takes off from the nose of one train at 20 mi/h, towards the other train. As soon as the fly reaches the other train, it turns around and heads off at 20 mi/h back towards the first train. It continues to do this until the trains collide.
> Question: How far does the fly fly before the collision?

There is a relatively easy solution: The trains will collide after exactly 1 hour (since they will each have gone 10 miles by then). Since the speed of the fly is 20 mi/h, it will have flown 20 miles by the time the trains collide. (This is sometimes described as the physicists solution).

Another solution goes as follows: The fly is twice as fast as the trains, so on the first leg of its flight, it will cover x_1 miles, while the train going in the opposite direction will have covered $x_1/2$ miles. The total, $x_1 + \frac{x_1}{2}$ must equal the initial distance: 20 miles. Thus

$$3x_1/2 = 20 \Longrightarrow x_1 = \frac{2}{3}20.$$

For the second leg of the flight, the new distance between the trains is $20 - 2\frac{x_1}{2} = 20 - \frac{2}{3}20 = \frac{1}{3}20$ miles, so by the same argument the fly will cover

$$x_2 = \frac{2}{3}(\frac{1}{3}20) \text{ miles.}$$

Similarly, the distance of the third leg is

$$x_3 = \frac{2}{3}(\frac{1}{3}20 - 2\frac{x_2}{2}) = \frac{2}{3}\frac{1}{3}(\frac{1}{3}20).$$

In general, the nth distance the fly covers is

$$x_n = \frac{2}{3}20(\frac{1}{3})^{n-1}.$$

Continuing this way infinitely many steps, and summing all the distances, we get the total distance the fly covers:

$$\begin{aligned} x_1 + x_2 + \cdots = &\frac{2}{3}20 + \frac{2}{3}20\frac{1}{3} + \frac{2}{3}20(\frac{1}{3})^2 + \cdots \\ &= \frac{2}{3}20\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots\right). \end{aligned}$$

We now need to evaluate the infinite series in the brackets. Let

$$x = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots.$$

[1] The story involves the mathematician John von Neumann. It is not examinable.

Then

$$3(x-1) = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots = x.$$

Solving for x, we get $x = 3/2$, so the solution to the original problem is $\frac{2}{3}20 \cdot x = 20$ miles, which luckily agrees with the first solution!

It is said that physicists instictively solve this problem in the first way, while mathematicians try to solve it using an infinite series. This may just be a myth, but it begs the question: Why do we even consider the more complicated solution? The answer is that many problems will not have an easy solution, so that the techniques of infinite series *has* to be used, and there is no alternative. That's why we study them.

Terminology

An individual piece of a series is called a *term*, often denoted by a subscript. For example, in the series

$$x_1 + x_2 + \cdots,$$

each x_i is a term, for $i = 1, 2, \ldots$.

There are *finite series*

$$x_1 + x_2 + \cdots + x_N = \sum_{i=1}^{N} x_i,$$

and *infinite series* (with infinitely many terms)

$$x_1 + x_2 + \cdots = \sum_{i=1}^{\infty} x_i.$$

Example 2.1. In the fly problem, we had the infinite series

$$1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots = \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i$$

(note that i starts from 0). The *sum* of this series is its value, which we computed to be 3/2.

If we write this series as $\sum_{i=1}^{\infty} x_i$, with $x_i = \left(\frac{1}{3}\right)^i$, then we have a *recurrence relation* $x_{i+1} = \frac{1}{3}x_i$, for each $i \geq 0$. A recurrence relation is a rule for computing a term from some of the previous terms.

Our goals are:

- To understand how to compute finite series $\sum_{i=1}^{N} x_i$.
- To define and investigate infinite series. Does the sum exist? If so, how do we compute it?

Series basics

Definition 2.2. Let $s = \sum_{i=1}^{\infty} a_i$ be an infinite series. A *partial sum* is

$$a_1 + a_2 + \cdots + a_N = \sum_{i=1}^{N} a_i = s_N.$$

In this way, we get a *sequence* of partials sums: $s_1, s_2, \ldots$. A series can also start at $i = 0$ or any other integer, even negative ones.

The series $s = \sum_{i=1}^{\infty} a_i$ *converges* if the sequence s_1, s_2 has a limit, that is if $\lim_{N\to\infty} s_n$ exists. This limit is then denoted by s, and it is the sum of the series. If the series does not converge, it is said to *diverge*.

Example 2.3.

- $1+0+0+0+\cdots$ converges. The sequence of partial sums $s_1, s_2, \ldots$ is $1, 1, 1, \ldots$ whose limit is just 1.
- $1+1+1+\cdots$ diverges. The sequence of partial sums $s_1, s_2, \ldots$ is $1, 2, 3, \ldots$, which does not have a limit.

Example 2.4. Arithmetic series (finite). This is where the difference between two consecutive terms is constant (e.g., $0+2+4+6+\cdots$ has constant difference 2). Therefore, we can write an arithmetic series as

$$(a+d) + (a+2d) + \cdots + (a+Nd) = s_N = \sum_{i=1}^{N} a_i, \text{ where } a_i = a + id.$$

where d is the difference. How do we compute these? Write the series once and once in reverse order:

$$\begin{aligned} s_N &= (a+d) + (a+2d) + \cdots + (a+Nd) \\ s_N &= (a+Nd) + (a+(N-1)d) + \cdots + (a+d). \end{aligned}$$

The sum of two terms lying above each other is always $2a + (N+1)d$, so

$$\begin{aligned} 2s_N &= \underbrace{(a+d) + (a+Nd) + \cdots + (a+Nd) + (a+d)}_{N \text{ pairs of terms}} \\ &= N(2a + (N+1)d). \end{aligned}$$

Thus $s_N = \frac{N}{2}(2a + (N+1)d)$. If $a = 0$ and $d = 1$, we get

$$1 + 2 + \cdots + N = \frac{N(N+1)}{2}.$$

Example 2.5. Geometric series. Here the quotient of two consecutive terms is constant (e.g., $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ with constant quotient $1/2$). They look like

$$s_N = \sum_{i=0}^{N} ar^i = a + ar + ar^2 + \cdots + ar^N$$

and we will always assume that $r \neq 1$ (if $r = 1$ it is an arithmetic series with difference 0). How do we compute these? If we multiply the series by r and add a we get

$$rs_N + a = a + r(a + ar + \cdots + ar^N) = a + ar + ar^2 + \cdots + ar^{N+1} = s_N + ar^{N+1}.$$

Solving for s_N, we get

$$s_N = a\frac{r^{N+1} - 1}{r - 1}.$$

If instead we consider the infinite series

$$\sum_{i=0}^{\infty} ar^i,$$

then have just seen that the partial sums are s_N as above. Now,

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} a\frac{r^{N+1} - 1}{r - 1} = a\frac{1}{1 - r} + \lim_{N\to\infty} a\frac{r^{N+1}}{r - 1} = \begin{cases} 0 & \text{if } a = 0, \\ a\frac{1}{1-r} & \text{if } |r| < 1. \end{cases}$$

This is because if $|r| < 1$, then

$$\lim_{N\to\infty} a\frac{r^{N+1}}{r - 1} = \frac{a}{r - 1} \lim_{N\to\infty} r^{N+1} = 0.$$

If $|r| \geq 1, a \neq 0$, then there is no limit. In particular, we have

$$1 + r + r^2 + \cdots = \frac{1}{1 - r}$$

whenever $|r| < 1$. We will use this important identity later.