Lecture 20

Bases

Let V be a vector space (e.g., $\mathbb{R}^n$ or $\mathbb{C}^n$). The span of some vectors in V is also a vector space, which can be all of V or smaller.

Definition 20.1. A *basis* of V is a set of vectors $\underline{v}_1, \ldots, \underline{v}_m$ such that:

i) this set is linearly independent,

ii) $\operatorname{span}\{\underline{v}_1, \ldots, \underline{v}_m\} = V$.

Example 20.2. The vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

form a basis for $\mathbb{R}^2$ (or $\mathbb{C}^2$), called the *standard basis*:

- They span all of $\mathbb{R}^2$ (or $\mathbb{C}^2$): any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can be written as

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- They are linearly independent: If $\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{0}$, then $\lambda_1 = \lambda_2 = 0$.

There are other bases for $\mathbb{R}^2$, for example

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

from Example (19.7).

On the other hand, the three vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

also span $\mathbb{R}^2$, but they are *not* linearly independent:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so these three vector do not form a basis.

Theorem 20.3. *Every vector space has a basis. For a given vector space V, the number of elements in a basis (if finite) is always the same. This number is called the* dimension *of V (notation:* $\dim V$*).*

For example, $\mathbb{R}^2$ (or $\mathbb{C}^2$) has dimension two. In $\mathbb{R}^3$ (or $\mathbb{C}^3$) we have the standard basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

so these spaces have dimension 3. More generally, the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are n-dimensional.

Matrices as linear maps

Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be a square matrix of size n. We can define a function from the vector space $\mathbb{R}^n$ to itself:

$$L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad L_A(\underline{x}) = A\underline{x}.$$

This map is compatible with addition and scalar multiplication, that is,

$$L_A(\underline{x}+\underline{y}) = A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y} = L_A(\underline{x}) + L_A(\underline{y})$$

and

$$L_A(\lambda\underline{x}) = A(\lambda\underline{x}) = \lambda A\underline{x} = \lambda L_A(\underline{x}), \qquad \lambda \in \mathbb{R}.$$

A function satisfying these two properties is called a *linear map*.

So, from a matrix, we get a linear map. We can also go the other way:

> Given a linear map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, we can write down a matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, such that $L_A = f$.

The way to do this is the following. Let $\{\underline{v}_1, \ldots, \underline{v}_n\}$ be the standard basis of $\mathbb{R}^n$, and $\{\underline{u}_1, \ldots, \underline{u}_m\}$ is the standard basis in $\mathbb{R}^m$. Then every $f(\underline{v}_j)$ is a linear combination of vectors of $\{\underline{u}_1, \ldots, \underline{u}_m\}$, so we can write for every $j = 1, \ldots, n$

$$f(\underline{v}_j) = a_{1j}\underline{u}_1 + a_{2j}\underline{u}_2 + \cdots + a_{mj}\underline{u}_m = \sum_{i=1}^{m} a_{ij}\underline{u}_i.$$

Then $f = L_A$, where $A = (a_{ij})$.

Here is an example to show how this is done.

Example 20.4. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the function

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -y \\ x \end{pmatrix}.$$

We first show that f is a linear map:

- Additivity:
$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = f\begin{pmatrix} x+a \\ y+b \\ z+c \end{pmatrix} = \begin{pmatrix} z+c \\ -(y+b) \\ x+a \end{pmatrix} = \begin{pmatrix} z \\ -y \\ x \end{pmatrix} + \begin{pmatrix} c \\ -b \\ a \end{pmatrix} = f\begin{pmatrix} x \\ y \\ z \end{pmatrix} + f\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$
- Scalar multiplication: $f\left(\lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \lambda z \\ -\lambda y \\ \lambda x \end{pmatrix} = \lambda f\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

We will now find a matrix A such that $L_A = f$. To do this, we choose the standard basis of $\mathbb{R}^n$.:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

To find the matrix with respect to this basis, we evaluate f on each basis vector:

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix}$$
$$f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

We then put the three resulting vectors together as the columns of a matrix:

$$A = \begin{pmatrix}0 & 0 & 1\\0 & -1 & 0\\1 & 0 & 0\end{pmatrix}.$$

Then our original map f equals L_A:

$$L_A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\0 & -1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\-y\\x\end{pmatrix} = f\begin{pmatrix}x\\y\\z\end{pmatrix}.$$