Lecture 21

Kernel, image and rank

Let $A \in \mathrm{Mat}_{m\times n}$, and $L_A : V \to W$ be a linear map. Here $\dim V = n$, $\dim W = m$.

Definition 21.1.
- A *kernel* of L_A is the set $\ker L_A = \{\underline{v} \in V \mid L_A(\underline{v}) = \underline{0}\}$.
- An *image* of L_A is the set $\mathrm{im}\, A = \{\underline{w} \in W \mid \underline{w} = L_A(\underline{v}) \text{ for some } \underline{v} \in V\}$.
- Recall that a kernel of a matrix A is the set of solutions of the homogeneous system $A\underline{x} = \underline{0}$. Thus, the kernel of A is precisely the kernel of L_A.

Example 21.2.
- $L_A = 0 : V \to W$, $\underline{v} \mapsto \underline{0} \in W$. Then $\ker L_A = V$, $\mathrm{im}\, L_A = \underline{0} \in W$.
- $L_A = \mathrm{id} : V \to V$, $\underline{v} \mapsto \underline{v}$ (*identity map*). Then $\ker L_A = \underline{0}$, $\mathrm{im}\, L_A = V$.
- $L_A : \mathbb{R}^2 \to \mathbb{R}$, $L_A \begin{pmatrix} x \\ y \end{pmatrix} = x$. Then $\ker L_A = \{\begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{R}\}$, $\mathrm{im}\, L_A = \mathbb{R}$.

Note that both the image and the kernel of a linear map are vector spaces themselves.

Definition 21.3. A *rank* of a linear map L_A is the dimension of its image. Equivalently, it is the maximal number of linearly independent columns in the corresponding matrix A. The latter is called a *rank* of A, notation: $\mathrm{rk}\, A$.

Example 21.4. $\mathrm{rk}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$; $\mathrm{rk}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1$; $\mathrm{rk}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$; $\mathrm{rk}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2$.

Fact. Let $A \in \mathrm{Mat}_{m\times n}$. Then
- $\mathrm{rk}\, A$ is also equal to the maximal number of linearly independent *rows* of A;
- therefore, $\mathrm{rk}\, A$ does not change under ERO;
- thus, the rank of A is actually equal to the number of non-zero rows in the RREF of A.

Example 21.5. Let $A = \begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 2 & 1 & 6 \end{pmatrix}$. The columns of A are the vectors $\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{v}_4 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$. Vectors $\underline{v}_1$ and $\underline{v}_2$ are linearly independent, but the other vectors are linear combinations of $\underline{v}_1$ and $\underline{v}_2$: $\underline{v}_3 = \underline{v}_2 - \underline{v}_1$, $\underline{v}_4 = 2\underline{v}_2$. Therefore, the maximal number of linear independent columns is 2, so $\mathrm{rk}\, A = 2$.

On the other hand, we can compute RREF of A:

$$\begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 2 & 1 & 6 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 1 & 6 \\ 0 & 1 & 1 & 3 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

Thus, the RREF of A has two non-zero rows, so we see again that $\mathrm{rk}\, A = 2$.

Let now $A \in \mathrm{Mat}_n$. We say that A is *singular* if $\mathrm{rk}\, A < n$, and *non-singular* otherwise. As we can see from the fact above, the rank of a square matrix A is equal to n if and only if the RREF of A is the identity matrix I_n, which, as we know (see Proposition 17.3), is the same as $\det A \neq 0$. Therefore, in view of Corollary 17.4, we can summarize this as follows:

Corollary 21.6. *For $A \in \mathrm{Mat}_n$ the following are equivalent:*

- $\operatorname{rk} A = n$ *(i.e., A is non-singular);*
- $\ker A = \underline{0}$ *(i.e. the homogeneous system $A\underline{x} = \underline{0}$ has a unique solution);*
- $\det A \neq 0$*;*
- *RREF of A is the identity matrix I_n;*
- *A is invertible.*

There is another way to find the rank of a matrix $A \in \mathrm{Mat}_{m\times n}$ based on the following property.

Fact. The rank of a matrix is equal to the maximal size of a non-singular square submatrix.

Example 21.7. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Since there are two rows only, we see that $\operatorname{rk} A \leq 2$. We observe that $\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$, so there is a 2×2 non-singular submatrix, which implies that $\operatorname{rk} A \geq 2$. Therefore, $\operatorname{rk} A = 2$.

Example 21.8. Let $A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & -4 \\ 0 & -3 & 6 \end{pmatrix}$. Observe that $\det A = 0$, so $\operatorname{rk} A < 3$. Further, any 2×2 submatrix of A is singular, so $\operatorname{rk} A < 2$. There are non-zero entries in A, which implies that $\operatorname{rk} A \geq 1$. Thus, $\operatorname{rk} A = 1$.