Lecture 22

One more fact about the rank.

Definition 22.1. Let $A \in \mathrm{Mat}_{m\times n}$. The dimension of the kernel of A is called *nullity* of A, notation $\operatorname{null} A$.

Nullity is closely related to rank:

Proposition 22.2. *For* $A \in \mathrm{Mat}_{m\times n}$*, we have* $\operatorname{null} A = n - \operatorname{rk} A$.

The reason for this is the following: the kernel is the set of solutions of the homogeneous system $A\underline{x} = \underline{0}$, so the dimension of the kernel is equal to the number of "free parameters" in the solution of the system. The rank is equal to the number of non-zero rows in RREF, so it is equal to the number of "non-free" variables. Therefore, these two numbers sum up to the number of variables, i.e. to n.

Application to linear ODEs

Ordinary Differential Equations (ODEs) come up in the modelling of engineering and physical problems. We can use matrices to help solve linear ODEs:

Example 22.3. Solve the ODE

$$y'' - 5y' + 4y = 0,$$

where $y = y(t)$ is a function in t, with the initial conditions $y(3) = 6$, $y'(3) = -1$.
Solution: We can write higher order ODE as a system with a change of variables. Let

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t). \end{aligned}$$

Taking derivatives, we get

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -4y + 5y' = -4x_1 + 5x_2. \end{aligned}$$

The initial conditions become

$$x_1(3) = 6, \qquad x_2(3) = -1.$$

Our ODE is thus rewritten as

$$\underline{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\underline{x},$$

where A is the 2×2 matrix.

Now if we had a one-variable ODE $x' = ax$, for $a \in \mathbb{R}$, then the solution would be $x(t) = ce^{at}$, for some constant c. For our equation $\underline{x}' = A\underline{x}$, let's see when

$$\underline{x} = \underline{b}e^{rt}$$

is a solution, for some vector $\underline{b}$ and $r \in \mathbb{R}$. Well, this will be a solution precisely when

$$\underline{x}' = \underline{b}re^{rt} = A\underline{b}e^{rt}.$$

Cancelling the e^{rt} (which are never zero!), we get

$$A\underline{b} = r\underline{b}.$$

So, we need to find the vectors $\underline{b}$ satisfying this. Such vectors are called *eigenvectors* of A with *eigenvalue* r.

To find these, we do the following:

- Compute the determinant of the matrix $A - \lambda I_2$:

$$\begin{aligned}\det(A-\lambda I_2) &= \left|\begin{pmatrix}0 & 1\\ -4 & 5\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right| = \left|\begin{pmatrix}-\lambda & 1\\ -4 & 5-\lambda\end{pmatrix}\right| = -\lambda(5-\lambda)+4\\ &= \lambda^2 - 5\lambda + 4.\end{aligned}$$

  Now find the roots of this polynomial:

$$\lambda_1 = 1, \quad \lambda_2 = 4.$$

  These are the eigenvalues of A.

- Next, solve the equation

$$A\underline{b} = r\underline{b}$$

  for each of the eigenvalues. For the first one:

$$\begin{pmatrix}0 & 1\\ -4 & 5\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix} = 1\cdot\begin{pmatrix}b_1\\ b_2\end{pmatrix} \iff \begin{pmatrix}b_2\\ -4b_1+5b_2\end{pmatrix} = \begin{pmatrix}b_1\\ b_2\end{pmatrix}$$
$$\iff \begin{cases}b_2 & = b_1\\ -4b_1+5b_2 & = b_2\end{cases} \iff b_2 = b_1.$$

  We only need one solution, for example

$$\underline{b} = \begin{pmatrix}b_1\\ b_2\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

  For the second eigenvalue, we similarly get a solution

$$\underline{b} = \begin{pmatrix}b_1\\ b_2\end{pmatrix} = \begin{pmatrix}1\\ 4\end{pmatrix}.$$

We now return to our system of ODEs: $\underline{x}' = A\underline{x}$, and see that we have found two solutions to it:

$$\underline{v}_1 = \begin{pmatrix}1\\ 1\end{pmatrix}e^{t}, \qquad \underline{v}_2 = \begin{pmatrix}1\\ 4\end{pmatrix}e^{4t}.$$

We now finish by using the following fact:

Fact. The set of solutions $\underline{x}$ (which are functions in t) of the system $\underline{x}' = A\underline{x}$ form a vector space. In fact, this space equals

$$\operatorname{span}\{\underline{v}_1, \underline{v}_2\}.$$

Thus, any solution of $\underline{x}' = A\underline{x}$ is a linear combination of $\underline{v}_1$ and $\underline{v}_2$, that is, the general solution is

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\underline{v}_1 + c_2\underline{v}_2,$$

for some $c_1, c_2 \in \mathbb{R}$.

Plugging in our initial values

$$x_1(3) = 6, \qquad x_2(3) = -1,$$

we can find the constants:

$$\begin{aligned} \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} &= \begin{pmatrix} 6 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^3 + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4\cdot 3}, \\ &\iff \begin{cases} c_1e^3 + c_2e^{12} &= 6 \\ c_1e^3 + 4c_2e^{12} &= -1 \end{cases}. \end{aligned}$$

Solving this, we get

$$c_1 = \frac{25}{3e^3}, \qquad c_2 = \frac{-7}{3e^{12}}.$$

So, the solution to our original equation is

$$y(t) = x_1(t) = c_1e^t + c_2e^{4t} = \frac{25}{3e^3}e^t + \frac{-7}{3e^{12}}e^{4t}.$$

The above example shows that it is of interest of find eigenvalues and eigenvectors of matrices.

Example 22.4. Let

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}.$$

Find the eigenvalues: $\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{pmatrix} = (3-\lambda)(-\lambda) + 2 = \lambda^2 - 3\lambda + 2$. Find the roots:

$$\lambda = 1, \qquad \lambda = 2.$$

Now find eigenvectors for each eigenvalue:

$$\begin{aligned} \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} = 0 \iff \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = 0 \\ \iff 2x + y = 0, \end{aligned}$$

so one eigenvector is, for example,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Note that eigenvectors are always defined up to scaling.
For the eigenvalue $\lambda = 2$ we similarly get an eigenvector

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus we have

$$\begin{aligned} A\begin{pmatrix} 1 \\ -2 \end{pmatrix} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ A\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}. \end{aligned}$$

That is, the linear map L_A fixes one eigenvector, and doubles the other.