Lecture 23

Eigenvalues and eigenvectors

Definition 23.1. Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Recall that if

$$A\underline{x} = \lambda \underline{x},$$

for some *non-zero* vector $\underline{x} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, then λ is called an *eigenvalue* of A and $\underline{x}$ is called an *eigenvector* of A (for the eigenvalue λ).

To find eigenvalues and eigenvectors, we proceed as follows:
Rewrite the equation $A\underline{x} = \lambda \underline{x}$ as

$$A\underline{x} - \lambda \underline{x} = (A - \lambda I)\underline{x} = \underline{0}.$$

We thus have a homogeneous system of linear equations, with coefficient matrix $A - \lambda I$. A linear system has either zero solutions, one solution or infinitely many solutions. A homogeneous system always has at least one solution: $\underline{x} = 0$, so the first possibility is excluded.

Now, if the determinant $\det(A - \lambda I)$ is not zero, then we know that $A - \lambda I$ has an inverse, so we would get *exactly one* solution

$$(A - \lambda I)^{-1}(A - \lambda I)\underline{x} = 0 \Longrightarrow \underline{x} = \underline{0}.$$

But an eigenvector is not allowed to be 0, so we will ignore this case. Thus, the only possibility is that

$$\det(A - \lambda I) = 0,$$

and for any λ satisfying this, we will have infinitely many solutions $\underline{x}$.

The LHS here will be a polynomial in λ of degree n; compare how in (22.3) we got

$$\det(A - \lambda I_2) = \lambda^2 - 5\lambda + 4.$$

The polynomial $\det(A - \lambda I)$ is called the *characteristic polynomial* of A. Its roots are those values of λ for which the equation $A\underline{x} = \lambda \underline{x}$ has a non-zero solution, so these roots are the eigenvalues of A.

Suppose now that a is an eigenvalue of A. To find the corresponding eigenvector(s), we solve the linear system

$$A\underline{x} = a\underline{x},$$

just like we did in (22.3). Note: We will have infinitely many eigenvectors.

We have seen examples at the last lecture, here are some other examples:

Example 23.2. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The characteristic polynomial is $\det(A - \lambda I) = 1 + \lambda^2$, so the eigenvalues are $\pm i$. The eigenvectors $\underline{v}_1$ and $\underline{v}_2$ can be found by solving the equations $(A - iI)\underline{v}_1 = \underline{0}$ and $(A - iI)\underline{v}_2 = \underline{0}$, so we can choose eigenvectors $\underline{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Example 23.3. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The characteristic polynomial is $\det(A - \lambda I) = (-\lambda)^2 = \lambda^2$, so there is only *one* eigenvalue $\lambda = 0$. The space of eigenvectors E_0 is given by

$$E_0 = \{\underline{x} \in \mathbb{C}^2 \mid A\underline{x} = \underline{0}\},$$

that is,

$$E_0 = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{C}\}.$$

Since E_0 is spanned by one vector, for example $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, it means that E_0 is a one-dimensional vector space. This means that any two vectors is E_0 are linearly dependent. Thus, there are no two vectors in E_0 which form a basis for $\mathbb{C}^2$.

Definition 23.4. The set of eigenvectors for an eigenvalue λ is called an *eigenspace*, denoted E_λ. Thus,

$$E_\lambda = \{\underline{v} \mid (A - \lambda I)\underline{v} = \underline{0}\}.$$

Example 23.5. Let $A = \begin{pmatrix} 1 & 0 & 6 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix}$. Determine the eigenvalues and eigenspaces of A.
Solution: For the eigenvalues:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 6 \\ 3 & 2-\lambda & 1 \\ 2 & 0 & 2-\lambda \end{vmatrix} = \text{(expand along the middle column)} \\
&= (2-\lambda) \begin{vmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)((1-\lambda)(2-\lambda) - 12) \\
&= (2-\lambda)(\lambda^2 - 3\lambda - 10) = (2-\lambda)(\lambda - 5)(\lambda + 2).
\end{aligned}$$

The last step is given by finding the roots of the quadratic polynomial. Thus the eigenvalues are

$$2, \quad 5, \quad -2.$$

The eigenvectors in the eigenspace E_2 are given by

$$A\underline{x} = 2\underline{x} \iff (A - 2I)\underline{x} = \underline{0} \iff \begin{pmatrix} -1 & 0 & 6 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} \underline{x} = 0.$$

Gauss elimination gives the equivalent system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = 0 \iff \underline{x} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, \text{ for any } y \in \mathbb{C}.$$

Thus

$$E_2 = \left\{ \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \mid y \in \mathbb{C} \right\}.$$

Similarly, for the eigenvalue 5, we get

$$\begin{pmatrix} -4 & 0 & 6 \\ 3 & -3 & 1 \\ 2 & 0 & -3 \end{pmatrix} \underline{x} = 0 \iff \begin{pmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -11/6 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = 0$$

$$\iff \underline{x} = \begin{pmatrix} 3z/2 \\ 11z/6 \\ z \end{pmatrix}, \; z \in \mathbb{C}.$$

$$\iff E_5 = \left\{ \begin{pmatrix} 3z/2 \\ 11z/6 \\ z \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

Finally, for -2, we get

$$\begin{pmatrix} 3 & 0 & 6 \\ 3 & 4 & 1 \\ 2 & 0 & 4 \end{pmatrix} \underline{x} = 0 \iff \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = 0$$

$$\iff \underline{x} = \begin{pmatrix} -2z \\ 5z/4 \\ z \end{pmatrix}, \; z \in \mathbb{C}.$$

$$\iff E_{-2} = \left\{ \begin{pmatrix} -2z \\ 5z/4 \\ z \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

We see that each of the eigenspaces are one-dimensional. Indeed, we can choose a one-element basis in each:

$$E_2 = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \qquad E_5 = \operatorname{span}\left\{ \begin{pmatrix} 9 \\ 11 \\ 6 \end{pmatrix} \right\}, \qquad E_{-2} = \operatorname{span}\left\{ \begin{pmatrix} 8 \\ -5 \\ -4 \end{pmatrix} \right\}.$$

Example 23.6. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$. Determine the eigenvalues and eigenspaces of A.

Solution: For the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{vmatrix} = (1-\lambda)^2(4+\lambda^2) = (1-\lambda)^2(2i+\lambda)(2i-\lambda).$$

Thus the eigenvalues are

$$\lambda_1 = 2i, \; \lambda_2 = -2i, \lambda_3 = 1.$$

The eigenvectors in the eigenspace E_{2i} are given by

$$\underline{0} = (A - 2iI)\underline{x} = \begin{pmatrix} 1-2i & 0 & 0 & 0 \\ 0 & 1-2i & 0 & 0 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{pmatrix} \underline{v}_1,$$

which is equivalent to

$$\underline{v}_1 = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix}, \ c_1 \in \mathbb{C}.$$

Thus

$$E_{2i} = \left\{ \begin{pmatrix} 0 \\ 0 \\ c_1 \\ ic_1 \end{pmatrix} \mid c_1 \in \mathbb{C} \right\}.$$

Similarly, for the eigenvalue $\lambda_2 = -2i$, we get

$$\underline{0} = (A - 2iI)\underline{x} = \begin{pmatrix} 1+2i & 0 & 0 & 0 \\ 0 & 1+2i & 0 & 0 \\ 0 & 0 & 2i & 2 \\ 0 & 0 & -2 & 2i \end{pmatrix} \underline{v}_2,$$

which is equivalent to

$$\underline{v}_2 = c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix}, \ c_2 \in \mathbb{C}.$$

Thus

$$E_{-2i} = \left\{ \begin{pmatrix} 0 \\ 0 \\ c_2 \\ -ic_2 \end{pmatrix} \mid c_2 \in \mathbb{C} \right\}.$$

Finally, for $\lambda_3 = 1$, we get

$$\underline{0} = (A - I)\underline{x} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix} \underline{v},$$

which is equivalent to

$$\underline{v} \in E_1 = \left\{ \begin{pmatrix} c_3 \\ c_4 \\ 0 \\ 0 \end{pmatrix} \mid c_3, c_4 \in \mathbb{C} \right\}.$$

We see that E_1 can be written as a span of the two linearly independent vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, so it is two-dimensional. Note also that we could choose a different basis in E_1, and thus obtain a different basis of $\mathbb{C}^4$ consisting of eigenvectors of A. We will come back to this in the process of diagonalization of matrices.