Lecture 24

Eigenvalues of special matrices

We now look at eigenvalues of matrices of special types.

Hermitian matrices. Let $A \in \mathrm{Mat}_n$ be Hermitian, so that $A^\dagger = A$. Let $\underline{v}$ be an eigenvector of A, $A\underline{v} = \lambda\underline{v}$. Let us compute $\underline{v}^\dagger A\underline{v}$ in two ways. First,

$$\underline{v}^\dagger A\underline{v} = (\underline{v}^\dagger A)\underline{v} = (\underline{v}^\dagger A^\dagger)\underline{v} = (A\underline{v})^\dagger\underline{v} = (\lambda\underline{v})^\dagger\underline{v} = \underline{v}^\dagger\lambda^*\underline{v} = \lambda^*(\underline{v}^\dagger\underline{v}).$$

On the other hand,

$$\underline{v}^\dagger A\underline{v} = \underline{v}^\dagger(A\underline{v}) = \underline{v}^\dagger\lambda\underline{v} = \lambda(\underline{v}^\dagger\underline{v}).$$

Comparing the two expressions and using the fact $\underline{v}^\dagger\underline{v} \neq 0$, we see that $\lambda^* = \lambda$. Therefore, eigenvalues of Hermitian matrices are always real.

Example 24.1. Let $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$. Then the characteristic polynomial is $\det(A - \lambda I) = (1 - \lambda)(2 - \lambda) - 1 = \lambda^2 - 3\lambda + 1$ which has two real roots $\lambda = \frac{3\pm\sqrt{5}}{2}$.

As real symmetric matrices are Hermitian, we see that they also have real eigenvalues. Further, the computation above applied to an anti-Hermitian matrix would give $\lambda^* = -\lambda$, which implies that the eigenvalues are purely imaginary (i.e., their real part is zero). Summarizing, we have

Corollary 24.2.

- *Eigenvalues of Hermitian matrices are real.*
- *Eigenvalues of real symmetric matrices are real.*
- *Eigenvalues of anti-Hermitian matrices are purely imaginary. In particular, eigenvalues of real anti-symmetric matrices are purely imaginary.*

Unitary matrices. Now let $A \in \mathrm{Mat}_n$ be unitary, so that $A^\dagger A = I$. Let $\underline{v}$ be an eigenvector of A, $A\underline{v} = \lambda\underline{v}$. Let us compute $\underline{v}^\dagger\underline{v}$. We have

$$\underline{v}^\dagger\underline{v} = \underline{v}^\dagger I\underline{v} = \underline{v}^\dagger A^\dagger A\underline{v} = (A\underline{v})^\dagger(A\underline{v}) = (\lambda\underline{v})^\dagger(\lambda\underline{v}) = \underline{v}^\dagger\lambda^*\lambda\underline{v} = (\lambda\lambda^*)(\underline{v}^\dagger\underline{v}) = |\lambda|^2(\underline{v}^\dagger\underline{v}).$$

Therefore, $|\lambda|^2 = 1$, so $|\lambda| = 1$.

Corollary 24.3. *Eigenvalues of unitary and real orthogonal matrices have modulus* 1.

Example 24.4. Let $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Then $\lambda = \pm e^{i\varphi}$, so $|\lambda| = 1$.

Diagonalization

A matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ is said to be *diagonalizable* if we can choose a basis for $\mathbb{C}^n$ consisting of eigenvectors of L_A.

The term *diagonalization* means the following. Given a basis $\{\underline{v}_1, \ldots, \underline{v}_n\}$ for $\mathbb{C}^n$, we can create a *matrix* $B = (b_{ij})$ of L_A *with respect to this basis*: we can write for every $j = 1, \ldots, n$

$$L_A(\underline{v}_j) = b_{1j}\underline{v}_1 + b_{2j}\underline{v}_2 + \cdots + b_{nj}\underline{v}_n = \sum_{i=1}^{n} b_{ij}\underline{v}_i.$$

If the basis consists of eigenvectors of L_A, then $L_A\underline{v}_j = \lambda_j\underline{v}_j$, so the matrix B is *diagonal*.

Example 24.5. We have already seen in (22.4) that $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ has eigenvectors

$$\underline{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \ \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Therefore,

$$L_A\underline{v}_1 = 1 \cdot \underline{v}_1 + 0 \cdot \underline{v}_2, \quad L_A\underline{v}_2 = 0 \cdot \underline{v}_1 + 2 \cdot \underline{v}_2,$$

and thus L_A is given by the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is the basis $\{\underline{v}_1, \underline{v}_2\}$.