Lecture 25

Example 25.1 (Diagonalization of 2×2 matrices)**.** If we have a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and choose a basis $\underline{v}_1, \underline{v}_2$ for $\mathbb{C}^2$, then the matrix for L_A w.r.t. this basis is given as follows. We can write

$$\begin{aligned} L_A(\underline{v}_1) &= A\underline{v}_1 = \alpha_1 \cdot \underline{v}_1 + \alpha_2 \cdot \underline{v}_2, \\ L_A(\underline{v}_2) &= A\underline{v}_2 = \alpha_3 \cdot \underline{v}_1 + \alpha_4 \cdot \underline{v}_2 \end{aligned} \tag{*}$$

for some uniquely determined $\alpha_i \in \mathbb{C}$ (because $\underline{v}_1, \underline{v}_2$ is a basis, so any vector is a unique linear combination of $\underline{v}_1, \underline{v}_2$). Now, the matrix for L_A w.r.t. this basis is

$$\begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{pmatrix}.$$

So, if A is diagonalizable, it means that we can choose $\underline{v}_1, \underline{v}_2$ such that $\alpha_2 = \alpha_3 = 0$ (which means exactly that $\underline{v}_1$ ($\underline{v}_2$, resp.) is an eigenvector for A with eigenvalue α_1 (α_4, resp.).

Now, if A is diagonalizable, we can form a matrix P whose columns are the basis vectors $\underline{v}_1, \underline{v}_2$ (who are the eigenvectors of A according to our choice). That is, if we write

$$\underline{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} z \\ w \end{pmatrix},$$

for some coordinates x, y, z, w, then

$$P = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$$

and equations $(*)$ become

$$\begin{aligned} A \begin{pmatrix} x \\ y \end{pmatrix} = \alpha_1 \begin{pmatrix} x \\ y \end{pmatrix} &\iff \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \alpha_1 \begin{pmatrix} x \\ y \end{pmatrix} \\ A \begin{pmatrix} z \\ w \end{pmatrix} = \alpha_4 \begin{pmatrix} z \\ w \end{pmatrix} &\iff \begin{pmatrix} az + bw \\ cz + dw \end{pmatrix} = \alpha_4 \begin{pmatrix} z \\ w \end{pmatrix}, \end{aligned}$$

so

$$AP = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} ax + by & az + bw \\ cx + dy & cz + dw \end{pmatrix} = \begin{pmatrix} \alpha_1 x & \alpha_4 z \\ \alpha_1 y & \alpha_4 w \end{pmatrix} = \begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_4 \end{pmatrix} = P \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_4 \end{pmatrix}.$$

In fact, one can show that P has an inverse: indeed, the columns of P are linearly independent (since $\underline{v}_1, \underline{v}_2$ form a basis of $\mathbb{C}^2$), and thus the rank of P is equal to 2, which implies that $\det P \neq 0$ and thus P is invertible (see Corollary 21.6). Therefore, we can multiply by P^{-1} on both sides to obtain

$$P^{-1}AP = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_4 \end{pmatrix}.$$

Thus, we come to the following **algorithm of diagonalization** of 2×2 matrices.

Step 1: Compute charasteritic polynnomial of A and eigenvalues λ_1 and λ_2.

Step 2: Find two linearly independent eigenvectors $\underline{v}_1$ and $\underline{v}_2$ of A.

Step 3: Compose a matrix P (which is called a *transformation matrix* whose columns are $\underline{v}_1$ and $\underline{v}_2$. Then $P^{-1}AP$ is diagonal with diagonal entries λ_1 and λ_2.

Remark.

- In Step 1, the eigenvalues λ_1 and λ_2 may coincide.
- In Step 2, we may not be able to find two linearly independent eigenvectors $\underline{v}_1$ and $\underline{v}_2$ of A. Then the whole procedure fails, which means that the matrix is not diagonalizable.
- If $\lambda_1 \neq \lambda_2$ then eigenvectors $\underline{v}_1$ and $\underline{v}_2$ are linearly independent, so the matrix is diagonalizable. Indeed, assuming that $\underline{v}_1$ and $\underline{v}_2$ are linearly dependent we conclude that they belong to the same eigenspace, and thus have the same eigenvalue, which leads to a contradiction.

Example 25.2. $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Then the characteristic polynomial is $\det(A - \lambda I) = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = (\lambda - 2)\lambda$, so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. Solving the homogenuous systems $(A - \lambda_i I)\underline{v}_i = \underline{0}$, we find the corresponding eigenspaces, and then we choose one eigenvector from each: we can take, for example, $\underline{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Then we get a matrix $P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ whose columns are $\underline{v}_1$ and $\underline{v}_2$, and one can check that

$$P^{-1}AP = \left(\frac{1}{-2i} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

All of this works similarly for any $n \times n$ matrix for $n \geq 3$.

We summarize the argument from Example 25.1:

Proposition 25.3. *Let $A \in \mathrm{Mat}_n(\mathbb{C})$ be diagonalizable, i.e. there is a basis $\underline{v}_1, \ldots, \underline{v}_n$ for $\mathbb{C}^n$ such that each $\underline{v}_i$ is an eigenvector for A. If P is the matrix whose columns are the vectors $\underline{v}_1, \ldots, \underline{v}_n$, then*

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for the eigenvector $\underline{v}_i$.

The essential requirement in the above proposition is that the vectors $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent (they will then be a basis). If A has n *distinct* eigenvalues (as we had in (22.4) and (23.5)), then it will have n linearly independent eigenvectors, and hence be diagonalizable. This is because of the following fact:

Fact. Eigenvectors with distinct eigenvalues are linearly independent.

Corollary 25.4. *If $A \in \mathrm{Mat}_n(\mathbb{C})$ has n distinct eigenvalues, then A is diagonalizable.*

Indeed, by the definition of an eigenvalue, for every eigenvalue λ_i we can find a non-zero eigenvector $\underline{v}_i$. Due to the fact above, they all are linearly independent, and since there are n of them they compose a basis of $\mathbb{C}^n$.

Remark. According to the Fundamental Theorem of Algebra, any complex polynomial in one variable of degree n always has n roots (some of which may coincide). In particular, this can be applied to the characteristic polynomial of $A \in \mathrm{Mat}_n(\mathbb{C})$. In other words, if $\det(A - \lambda I)$ has k distinct roots, we can write

$$\det(A - \lambda I) = (\lambda_1 - \lambda)^{l_1}(\lambda_2 - \lambda)^{l_2} \ldots (\lambda_k - \lambda)^{l_k} = \Pi_{i=1}^{k}(\lambda_i - \lambda)^{l_i},$$

where the sum of all l_i is equal to n (numbers l_i are called *multiplicities* of roots λ_i).