# Lecture 26

We have seen that we can easily diagonalize a matrix having $n$ distinct eigenvalues. Even if the eigenvalues are not distinct, a matrix may still be diagonalizable:

**Example 26.1.** Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & -2 & 0 \end{pmatrix}$. Its eigenvalues are given by the roots of the polynomial

$$(1-\lambda)((-1-\lambda)(-\lambda)-2) = (1-\lambda)(\lambda^2+\lambda-2) = (1-\lambda)^2(\lambda+2),$$

that is, eigenvalues: $1, -2$.
We now find the eigenspaces. For $\lambda = 1$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & -1 \\ 1 & -2 & -1 \end{pmatrix} \underline{x} = 0 \Longleftrightarrow x - 2y - z = 0.$$

Thus, the eigenspace has two free parameters:

$$E_1 = \left\{ \begin{pmatrix} 2y+z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{C} \right\}.$$

Thus $E_1$ is a two-dimensional space, so we can find two linearly independent vectors in it, for example

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$E_1 = \left\{ y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{C} \right\} = \operatorname{span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Moreover, for $\lambda = -2$, we get

$$\begin{aligned} \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -2 & 2 \end{pmatrix} \underline{x} = 0 &\Longleftrightarrow \begin{pmatrix} 0 & 6 & -6 \\ 0 & 3 & -3 \\ 1 & -2 & 2 \end{pmatrix} \underline{x} = 0 \\ &\Longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = 0 \Longleftrightarrow \begin{cases} x & = 0 \\ y - z & = 0. \end{cases} \Longleftrightarrow \underline{x} = \begin{pmatrix} 0 \\ y \\ y \end{pmatrix}, \; y \in \mathbb{C}. \end{aligned}$$

Thus $E_{-2} = \left\{ \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \mid y \in \mathbb{C} \right\} = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Since $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for an eigenvalue which is distinct from that for the eigenvectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, we see that

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are linearly independent. Thus, they form a basis for $\mathbb{C}^3$ and hence $A$ is diagonalizable.

In fact, if we form the matrix

$$P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

then Proposition 25.3 tells us that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Summarizing, we just need to choose a basis in each eigenspace, and then collect all of these together to compose a basis of the whole space $\mathbb{C}^n$. This can be done in the only case when we have "enough" linearly independent eigenvectors in every eigenspace, i.e. if the sum of all of the dimensions $\dim E_{\lambda_i}$ is equal to $n$. In fact, the following (non-trivial) statement always holds.

**Fact.** Let $\det(A - \lambda I) = \Pi_{i=1}^{k}(\lambda_i - \lambda)^{l_i}$, where the sum of all $l_i$ is equal to $n$ (recall that numbers $l_i$ are called *multiplicities* of roots $\lambda_i$). Then for every $i = 1, \dots, k$ one has $\dim E_{\lambda_i} \leq l_i$.

Therefore, the example above can be generalized in the following way:

**Theorem 26.2.** *Let $A \in \mathrm{Mat}_n(\mathbb{C})$, let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of $A$, and let the characteristic polynomial of $A$ be $\det(A - \lambda I) = \Pi_{i=1}^{k}(\lambda_i - \lambda)^{l_i}$, where the sum of all $l_i$ is equal to $n$. Then $A$ is diagonalizable if and only if for every $i = 1, \dots, k$ the dimension of the eigenspace $E_{\lambda_i}$ is equal to $l_i$.*

Note that, by definition, $E_{\lambda_i} = \ker(A - \lambda_i I)$, and thus $\dim E_{\lambda_i} = n - \mathrm{rk}(A - \lambda_i I)$. Thus, Theorem 26.2 can be reformulated in the following easy-to-use way:

**Corollary 26.3.** *A matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ with eigenvalues $\lambda_1, \dots, \lambda_k$ and characteristic polynomial $\det(A - \lambda I) = \Pi_{i=1}^{k}(\lambda_i - \lambda)^{l_i}$ is diagonalizable if and only if for every $i = 1, \dots, k$ we have $n - \mathrm{rk}(A - \lambda_i I) = l_i$.*

Corollary 26.3 leads to the following algorithm.

**Criterion of diagonalizability of a matrix.** Let $A \in \mathrm{Mat}_n(\mathbb{C})$. To decide whether $A$ is diagonalizable, we need to do the following.

**Step 1**. Compute the characteristic polynomial $\det(A - \lambda I) = (\lambda_1 - \lambda)^{l_1}(\lambda_2 - \lambda)^{l_2} \dots (\lambda_k - \lambda)^{l_k}$.

**Step 2**. For every $i = 1, \dots, k$ compute the number $n - \mathrm{rk}(A - \lambda_i I)$.

**Step 3**. If for *every* $i = 1, \dots, k$ we have $n - \mathrm{rk}(A - \lambda_i I) = l_i$, then $A$ is diagonalizable. Otherwise, it is not.

*Remark.* In Steps 2 and 3, we need to consider only eigenvalues $\lambda_i$ with $l_i > 1$. Indeed, since we know that $1 \leq n - \mathrm{rk}(A - \lambda_i I) \leq l_i$, the equality $l_i = 1$ guarantees that $n - \mathrm{rk}(A - \lambda_i I) = 1 = l_i$ (in particular, we immediately get Corollary 25.4).