Lecture 27

Computing high powers of a diagonalizable matrix

A typical problem about matrices is: Given a matrix, for example, $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$, compute

$$A^{1000} = \underbrace{A \cdot A \cdots A}_{1000 \text{ times}}.$$

If we start by simply trying to multiply A with itself:

$$\begin{aligned} A^2 &= \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ -6 & -2 \end{pmatrix} \\ A^3 &= \cdots \end{aligned}$$

we quickly realize that this is going to be a lot of work, even for a computer. However, if A is diagonalizable, which we know that this A is (by (22.4)), it means that there is a matrix P such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix},$$

and it is easy to take a large power of a diagonal matrix:

$$(P^{-1}AP)^{1000} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}^{1000} = \begin{pmatrix} \lambda_1^{1000} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^{1000} \end{pmatrix}.$$

Now, the left hand side is

$$(P^{-1}AP)^{1000} = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) = P^{-1}A^{1000}P$$

(note that all the "inner" PP^{-1} cancel, because $PP^{-1} = I$). So

$$P^{-1}A^{1000}P = \begin{pmatrix} \lambda_1^{1000} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^{1000} \end{pmatrix},$$

and thus

$$A^{1000} = P\begin{pmatrix} \lambda_1^{1000} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^{1000} \end{pmatrix}P^{-1},$$

which we can compute, if we know P.

Similarly, we can compute a power series of a diagonalizable matrix A, e.g. exponent: if we denote the diagonal matrix $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$ as Λ, we have

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{(P\Lambda P^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{P\Lambda^k P^{-1}}{k!} = P(\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!})P^{-1} = P\exp(\Lambda)P^{-1},$$

where

$$\exp(\Lambda) = \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^k \end{pmatrix} = \begin{pmatrix} e_1^\lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_n^\lambda \end{pmatrix}.$$

Example 27.1. Let's take $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$, as above. By (22.4) we know that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

with $P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$. Thus, by the above discussion,

$$A^{1000} = P \begin{pmatrix} 1 & 0 \\ 0 & 2^{1000} \end{pmatrix} P^{-1}.$$

So to compute this, we need P^{-1}:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2+2R_1} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \end{array}\right),$$

so

$$P^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} A^{1000} &= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{1000} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^{1000} \\ -2 & -2^{1000} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{1001} - 1 & 2^{1000} - 1 \\ 2 - 2^{1001} & 2 - 2^{1000} \end{pmatrix}. \end{aligned}$$

The entries here are quite large numbers, whose decimal expansions could be calculated, but are best just left as they are.

Now,

$$\exp(A) = P \exp \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} e^1 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2e^2 - e & e^2 - e \\ 2e - e^2 & 2e + e^2 \end{pmatrix}.$$

Let's take a 3×3 as well:

Example 27.2. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & -2 & 0 \end{pmatrix}$. We will compute the nth power of A, for an arbitrary integer $n \geq 1$. By (26.1), we know that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

where

$$P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We need P^{-1}. Calculating it via Gauss elimination as before, gives

$$P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

Thus, for each n,

$$\begin{aligned} A^n &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-2)^n \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & (-2)^n \\ 0 & 1 & (-2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ 1-(-2)^n & 1+2(-2)^n & -1+2(-2)^n \\ 1-(-2)^n & -2+2(-2)^n & 2+(-2)^n \end{pmatrix}. \end{aligned}$$

Note that $(-2)^n = (-1)^n 2^n = \begin{cases} 2^n & \text{if } n \text{ is even} \\ -2^n & \text{if } n \text{ is odd} \end{cases}.$

Remark: It is only for diagonalizable matrices we can do this. For non-diagonalizable matrices the powers can also be calculated by using *Jordan normal form*.