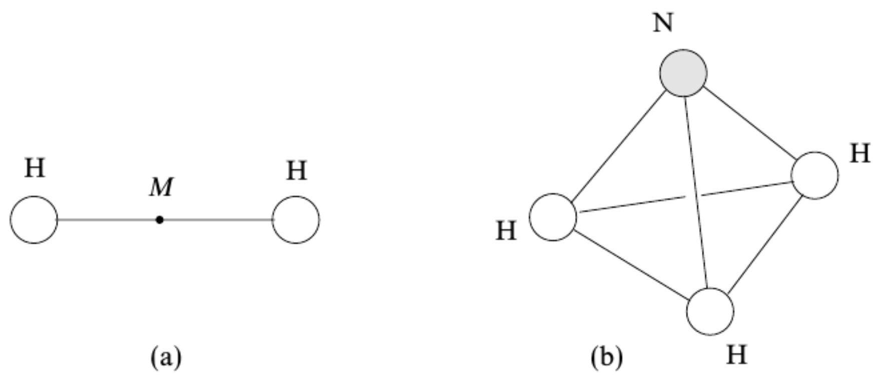
Single Maths A – Easter 2018

Lecture 1

1.1 Groups of symmetries

Groups are algebraic gadgets which are collections of symmetries of some objects. These can be symmetries of some mathematical object, or of a physical object, such as a molecule. Consider this picture of a hydrogen molecule (H_2) and an ammonia molecule (NH_3):



Mathematically, a *symmetry* is a transformation that carries an object into itself. For example, the hydrogen molecule (consisting of two hydrogen atoms) has the following symmetries:

i) Any rotation along its long axis,

ii) Rotation by π about an axis perpendicular to the long axis, and passing through M (which lies midway between the atoms). This symmetry can also be seen as a reflection (mirror image) w.r.t. the plane passing through M, perpendicular to the long axis.

iii) Any combination of the above.

The ammonia molecule is a little bit more complicated. It is a tetrahedron with a nitrogen atom at the top and a base which is an equilateral triangle with hydrogen atoms in the corners. Let A be the axis going through the N-vertex perpendicular to the base triangle. The symmetries of the ammonia molecule are then:

i) Rotations of $2\pi/3$, $4\pi/3$ or 2π about the axis A.

ii) Reflections in each of the three planes containing A and one of the hydrogen atoms.

iii) Any combination of the above.

Remark. If ammonia had another H atom instead of the N at the top, then it would have more symmetries. However, such a molecule doesn't exist in nature.

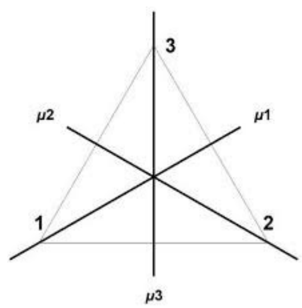
We note three important properties of these collections of symmetries:

- There is an identity symmetry (which brings us back to the initial position, for example rotation by 2π in the ammonia example).
- Composing two symmetries, that is, applying one followed by another, is again a symmetry.
- Every symmetry has an inverse ("do the opposite").

These three properties are the properties that define a *group*. We will give a few more examples, leading up to the precise mathematical definition of group.

1.2 Symmetries of regular n-gons in the plane

Consider an equilateral triangle (aka regular 3-gon):



The vertices are labeled $1, 2, 3$ and there are three reflection axes, μ_1, μ_2, μ_3. The symmetries of the triangle are then:

i) Rotations of $2\pi/3$, $4\pi/3$ or 2π about the centre point of the triangle (compare with the rotation symmetries of the ammonia molecule). Rotating anti-clockwise by $2\pi/3$ has the following effect on the vertices:

$$1 \longrightarrow 2, \qquad 2 \longrightarrow 3, \qquad 3 \longrightarrow 1,$$

that is, the vertices are cycled one step, so the triangle $(1, 2, 3)$ is transformed into $(3, 1, 2)$ (here we denote a triangle by starting from the lower left vertex and going anti-clockwise). Let's call this rotation r.
Rotating by $4\pi/3$ cycles two steps, that is,

$$(1, 2, 3) \longrightarrow (2, 3, 1).$$

This is the same thing as performing the rotation r twice. Viewing the rotation r as a function, the second rotation is $r \circ r = r^2$ (r composed with itself).
Finally, 2π, that is, rotating three steps, brings us back to the initial position, so this rotation is $r \circ r \circ r = r^3 = \mathrm{Id}$, the identity.

ii) Reflecting in the axis μ_1, the triangle $(1, 2, 3)$ is transformed into $(1, 3, 2)$; call this transformation s. Applying s twice is the identity, because the mirror image of a mirror image is the original image. Thus $s \circ s = s^2 = \mathrm{Id}$.
Reflecting in μ_2, is $(1, 2, 3) \longrightarrow (3, 2, 1)$; call this t.
Reflecting in μ_2, is $(1, 2, 3) \longrightarrow (2, 1, 3)$; call this u. As for any reflections, we have $t^2 = u^2 = \mathrm{Id}$.

iii) So far we have six symmetries, three rotations and three reflections:

$$\mathrm{Id}, r, r^2, s, t, u.$$

Are there any more? The only way we could get any more would be by combining a rotation with a reflection, or by combining two reflections (combining two rotations clearly gives another rotation).
If we perform the rotation r followed by the reflection s, we first get the triangle $(3, 1, 2)$, and then swap the second and third vertices by the reflection s, to get the triangle $(3, 2, 1)$. This has the same effect as the reflection t. Thus the composition is

$$s \circ r = sr = t.$$

On the other hand, if we first reflect by s and then rotate by r, we first get $(1, 3, 2)$ and then $(2, 1, 3)$. Thus

$$r \circ s = rs = u.$$

Thus we can generate the reflections t and u by forming combinations of only r and s.

Next, we check what happens if we combine r^2 with s. The symmetry r^2 gives $(2, 3, 1)$, and applying s to this gives $(2, 1, 3)$. Thus

$$sr^2 = u = rs.$$

Taking s first and then r^2 gives $(1, 3, 2)$, followed by $(3, 2, 1)$, so

$$r^2 s = t = sr.$$

Moreover, combining the identity with s, r or r^2 does not give anything new, that is,

$$\mathrm{Id}\, r = r\, \mathrm{Id} = r, \qquad \mathrm{Id}\, r^2 = r^2\, \mathrm{Id} = r^2, \qquad \mathrm{Id}\, s = s\, \mathrm{Id} = s.$$

Finally, perhaps the inverse of a symmetry (i.e., doing it backwards) would give something new? The inverse of a rotation (which for us is anti-clockwise) is just a clockwise rotation, and it is clear that going two steps forward (r^2) is the same as one step backwards (r^{-1}), that is,

$$r^{-1} = r^2.$$

Moreover, the inverse $s^{-1} = s$ (because to reverse the effect of s we just apply s again).
We therefore see that we indeed only have six symmetries in total:

$$D_3 = \{\mathrm{Id}, r, r^2, s, rs, r^2 s\}.$$

This is called the *dihedral group* D_3, or the symmetry group of the regular triangle.