Lecture 3

The difference method

This is a method which is sometimes useful to calculate finite or infinite series.

Example 3.1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Then the partial sums are $s_N = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{N(N+1)}$. Observe that

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

(partial fraction decomposition). Using this, we can write

$$s_N = \frac{1}{1} \underbrace{-\frac{1}{2} + \frac{1}{2}}_{=0} - \frac{1}{3} + \cdots + \frac{1}{N-1} \underbrace{-\frac{1}{N} + \frac{1}{N}}_{=0} - \frac{1}{N+1} = 1 - \frac{1}{N+1} = \frac{N}{N+1}.$$

In particular, letting $N \to \infty$, we get the sum of the infinite series: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Whenever we have a series $\sum_{n=1}^{\infty} a_n$ where we can write

$$a_n = f(n) - f(n+1)$$

for some function f (in the above, we had $f(n) = \frac{1}{n}$), then

$$s_N = f(1) - f(N+1)$$

and we can compute the sum of the infinite series as $\lim_{N\to\infty} f(1) - f(N+1)$ which is equal to $f(1) - \lim_{N\to\infty} f(N+1)$. This is called the *difference method* of computing the sum.

Here is an extension of the method when we jump more than one step each time:

Example 3.2. Let $s = \sum_{n=1}^{\infty} \frac{1}{n(n+3)}$. We can write

$$\frac{1}{n(n+3)} = \frac{1}{3}(\frac{1}{n} - \frac{1}{n+3}) = f(n) - f(n+3),$$

where $f(n) = \frac{1}{3n}$. Thus

$$\begin{aligned} s_N &= f(1) - f(4) + f(2) - f(5) + f(3) - f(6) + f(4) - f(7) + \cdots + \\ &\quad + f(N-2) - f(N+1) + f(N-1) - f(N+2) + f(N) - f(N+3) \\ &= f(1) + f(2) + f(3) - f(N+1) - f(N+2) - f(N+3). \end{aligned}$$

Letting $N \to \infty$, we is we get

$$s = \lim_{N\to\infty} s_N = f(1) + f(2) + f(3) = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18},$$

since $f(N) \to 0$ as $N \to \infty$ (so $f(N+1)$ etc vanish in the limit).

Here is another example using the difference method:

Example 3.3. Compute $s_N = \sum_{n=1}^{N} n^3$. We need to find a function f such that $n^3 = f(n) - f(n+1)$. First, note that if $f(n) = (n(n-1))^2$, then

$$f(n) - f(n+1) = -4n^3,$$

which isn't quite right. But if we modify this to cancel the -4 factor and take $g(n) = -\frac{1}{4}(n(n-1))^2$, we do get

$$g(n) - g(n+1) = n^3.$$

Then the difference method now says that $s_N = g(1) - g(N+1) = -\frac{1}{4}(0 - (N(N+1))^2) = \frac{(N(N+1))^2}{4}$.

Manipulating series

In order to simplify series and to be able to sum them it is sometimes useful to transform a series into another one, for example by differentiating or integrating it. In order to manipulate infinite series, we always need to ensure that the series converge.

Example 3.4. Let $s(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \cdots$. Differentiating, we get

$$\frac{ds}{dx} = \sum_{n=0}^{\infty} \left(\frac{x^{n+1}}{n+1}\right)' = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ if } |x| < 1 \text{ by (2.5).}$$

Integrating, we get $s(x) = \int \frac{1}{1-x} dx = -\ln(1-x) + c$. To determine the constant c, note that $s(0) = 0$, so we must have $c = 0$ and

$$s(x) = -\ln(1-x) \text{ for } |x| < 1.$$

Warning: Here is an example of how things can go wrong if the series does not converge. Let

$$s = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + \cdots.$$

If we multiply both sides by 2, we get

$$2s = 2 + 4 + 8 + \cdots = s - 1 \Longrightarrow s = -1,$$

which is clearly nonsense since we are summing positive numbers. Moral of the story: Do not manipulate diverging series. It is therefore important to know when a series converges, and this is what we will look at now.

Convergence, necessary conditions, tests

Proposition 3.5 (Necessary condition for convergence). *If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.*

In other words, if a_n does not go to zero, then the series diverges. This immediately tells us that the series $2 + 4 + 8 + \cdots$ diverges, and similarly $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots$ diverges. However, the converse of this proposition is not true.