Lecture 4

4.1 Maps between groups

Sometimes we can identify two groups as being essentially the same, but written in different ways. To make a connection between two groups, we need a function, or map, between them.

For example, the subgroup $\{\mathrm{Id}, r, r^2\} \subset D_3$ of rotations is essentially the same as the cyclic group $\mathbb{Z}/3$. Indeed, we have a map between them

$$\begin{aligned} \mathrm{Id} &\longmapsto 0, \\ r &\longmapsto 1, \\ r^2 &\longmapsto 2. \end{aligned}$$

This is a function $\varphi : \{\mathrm{Id}, r, r^2\} \to \mathbb{Z}/3$, which is clearly a bijection (i.e., one-to-one). But φ also has another important property, namely that the operation in $\{\mathrm{Id}, r, r^2\}$ (i.e., composition of rotations) is carried into the operation in $\mathbb{Z}/3$ (namely, addition mod 3). To check this, we write

$$\varphi(r \cdot r^2) = \varphi(\mathrm{Id}) = 0 = 1 + 2 \mod 3 = \varphi(r) + \varphi(r^2) \mod 3.$$

We should also check all the other possible products $\varphi(\mathrm{Id}\, r)$ and $\varphi(\mathrm{Id}\, r^2)$, but these are very simple because Id does not change the elements.

It is the bijection φ which carries one operation to the other, which makes these two groups have the same structure, and so be essentially the same. This is formulated in the following definition:

Definition 4.1. If G is a group with operation $*_G$ and H another group with operation $*_H$ then we say that G and H are *isomorphic* if there exists a bijection $\varphi : G \to H$ such that

$$\varphi(x *_G y) = \varphi(x) *_H \varphi(y),$$

for all $x, y \in G$.

We finish by giving another example of this:

Example 4.2. Let $G = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid 0 \le \theta < 2\pi \right\}$. We claim that G is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ and that it is isomorphic to the circle group S^1. Indeed, multiplying two such matrices

$$\begin{aligned} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} &= \begin{pmatrix} \cos\theta\cos\sigma - \sin\theta\sin\sigma & \cos\theta\sin\sigma + \sin\theta\cos\sigma \\ -\sin\theta\cos\sigma - \cos\theta\sin\sigma & -\sin\theta\sin\sigma + \cos\theta\cos\sigma \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\sigma) & \sin(\theta+\sigma) \\ -\sin(\theta+\sigma) & \cos(\theta+\sigma) \end{pmatrix} \in G. \end{aligned}$$

This also makes it clear that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so every element in G has an inverse in G. This is enough to make G a subgroup.

Now, consider the map

$$\varphi : G \longrightarrow S^1 = \{e^{i\theta} \mid 0 \le \theta < 2\pi\}$$

$$\varphi \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{i\theta}.$$

This is a bijection because it is clearly one-to-one. Finally, we have

$$\begin{aligned} \varphi\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix}\right) &= \varphi\begin{pmatrix} \cos(\theta+\sigma) & \sin(\theta+\sigma) \\ -\sin(\theta+\sigma) & \cos(\theta+\sigma) \end{pmatrix} = e^{i(\theta+\sigma)} \\ &= e^{i\theta}e^{i\sigma} = \varphi\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\varphi\begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix}. \end{aligned}$$

Thus φ is an isomorphism, and G is just another way of writing the circle group.