Lecture 4

Convergence, necessary conditions, tests

Proposition 3.5 (Necessary condition for convergence). *If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.*

In other words, if a_n does not go to zero, then the series diverges. This immediately tells us that the series $2 + 4 + 8 + \cdots$ diverges, and similarly $\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + \cdots$ diverges. However, the converse of this proposition is not true as the following example shows.

Example 4.1. (The harmonic series). The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series. We will show that it diverges. Some of the first partial sums are

$$s_1 = 1, \quad s_2 = 1 + \frac{1}{2} = \frac{3}{2}, \quad s_4 = s_2 + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2.$$

We can therefore guess that we always have

$$s_{2^n} \geq \frac{n}{2} + 1.$$

We now prove this by induction. Assume it is true for some $n \geq 1$. Now

$$\begin{aligned} s_{2^{n+1}} &= s_{2^n} + \frac{1}{2^n + 1} + \frac{1}{2^n + 2} + \cdots + \frac{1}{2^n + 2^n} \\ &> s_{2^n} + \frac{1}{2^n + 2^n} + \frac{1}{2^n + 2^n} + \cdots + \frac{1}{2^n + 2^n} \\ &= s_{2^n} + \frac{2^n}{2^{n+1}} \geq \frac{n}{2} + 1 + \frac{1}{2} \quad \text{(by the induction hypothesis)} \\ &= \frac{n+1}{2} + 1. \end{aligned}$$

This shows that $s_{2^n} \geq \frac{n}{2} + 1$, for all $n \geq 1$, and thus the partial sums tend to ∞ as $n \to 0$.

Assume now that all the terms in the series are ≥ 0. We have the following convergence tests:

Comparison test

Assume that $S = \sum_{n=0}^{\infty} b_n$ converges and $a_n \leq b_n$ for all n large enough. Then $s = \sum_{n=0}^{\infty} a_n$ converges. Reason: $s_N \leq S_N$ for each N.

Equivalently, if $s = \sum_{n=0}^{\infty} a_n$ diverges, and $a_n \leq b_n$, then $S = \sum_{n=0}^{\infty} b_n$ diverges.

Example 4.2. Let $s = \sum_{n=0}^{\infty} \frac{2^n + 3^n}{4^n}$. Then

$$a_n = \frac{2^n + 3^n}{4^n} \leq \frac{3^n + 3^n}{4^n} = 2\left(\frac{3}{4}\right)^n = b_n.$$

But $\sum_{n=0}^{\infty} b_n = 2 \sum_n \left(\frac{3}{4}\right)^n = \frac{2}{1 - 3/4}$ (geometric series with $|x| < 1$, see (2.5)).

Quotient test

Let $s = \sum_{n=0}^{\infty} a_n$ and $S = \sum_{n=0}^{\infty} b_n$ as before. Compute $p = \lim_{n\to\infty} \frac{a_n}{b_n}$. Then:

- if $p = 0$ and S converges, then s converges.
- if $p = \infty$ and s converges, then S converges.
- If $0 < p < \infty$, then s converges if and only if S does.
- If p doesn't exist (i.e., $\frac{a_n}{b_n}$ does not have a limit), then we can't say anything.

Example 4.3. Let $s = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_n a_n$ and $S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_n b_n$. Now

$$p = \lim \frac{a_n}{b_n} = \lim \frac{1/n^2}{\frac{1}{n(n+1)}} = \lim \frac{n+1}{n} = 1.$$

Thus, by the quotient test, s converges if and only if S does. But we know that S converges by (3.1), so we conclude that s also converges.

We can also use the comparison test to show that s converges: We have

$$s = \sum_{n=1}^{\infty} \frac{1}{n^2} = s = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \leq 1 + \sum_{n=0}^{\infty} \frac{1}{n(n+1)},$$

where we know that the last series converges, by (3.1). Thus the comparison test implies convergence also for s.

Moreover, $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges for any $k \geq 2$ because $\lim \frac{1/n^k}{1/n^2} = \lim \frac{1}{n^{k-2}} = 0$, for any $k > 2$.

As we saw in the above example, it is often possible to use several different tests to show convergence (but sometimes only one test works; it depends on the series).