Lecture 5

Ratio test

Let $s = \sum_{n=0}^{\infty} a_n$ and $p = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, if the limit exists. Then

- if $p < 1$, then s converges.
- if $p > 1$, then s diverges.
- if $p = 1$, then anything can happen (s may converge, may diverge).

Example 5.1. Let $s = \sum_{n=0}^{\infty} \frac{n^2}{2^n} = \sum a_n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{1}{2}\frac{(n+1)^2}{n^2} = \frac{1}{2}(1+\frac{1}{n})^2 \to \frac{1}{2} \text{ as } n \to \infty.$$

Since $\frac{1}{2} < 1$, we conclude that s converges.

Cauchy's root test

Let $s = \sum_{n=0}^{\infty} a_n$ and $p = \lim_{n\to\infty} \sqrt[n]{a_n}$, if the limit exists. Then

- if $p < 1$, then s converges.
- if $p > 1$, then s diverges.
- if $p = 1$ anything can happen.

Example 5.2. Does $s = \sum_{n=1}^{\infty} \frac{2n}{3^n}$ converge or diverge? The nth root of the nth term is

$$\left(\frac{2n}{3^n}\right)^{1/n} = \frac{(2n)^{1/n}}{3},$$

so we need to compute the limit of $\frac{(2n)^{1/n}}{3}$ as $n \to \infty$. Taking the logarithm of the numerator, we have

$$\ln((2n)^{1/n}) = \frac{\ln(2n)}{n},$$

and by l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\ln(2n)}{n} = \lim_{n\to\infty} \frac{2/2n}{1} = 0.$$

Exponentiating both sides, we get $\lim((2n)^{1/n}) = e^0 = 1$, so

$$\lim \frac{(2n)^{1/n}}{3} = \frac{1}{3} < 1.$$

Thus, by the root test, s converges.

Integral test

Suppose we can find a *decreasing* function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(n) = a_n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\lim_{N\to\infty} \int_1^N f(x)dx \quad \text{exists.}$$

The idea here is that the integral is a good approximation of the series, so one converges iff the other does.

Example 5.3. Let $s = \sum_{n=1}^{\infty} \frac{1}{n^q}$, $q > 0$. Then we can take $f(x) = \frac{1}{x^q}$, which is decreasing. We have, if $q \neq 1$:

$$\int_1^N f(x)dx = \left[\frac{1}{-q+1} x^{-q+1}\right]_1^N = \frac{1}{1-q}(N^{-q+1} - 1).$$

Thus:

- If $q > 1$, we get $\lim_{N\to\infty} \frac{1}{N^{q-1}} = 0$, so $\lim \int_1^N f(x)dx = \frac{1}{q-1}$, hence the series s converges.
- If $q < 1$, then $\lim_{N\to\infty} \frac{1}{N^{q-1}} = \infty$, so the integral has no limit, hence the series s diverges.

If $q = 1$, we have $f(x) = \frac{1}{x}$, so $\int_1^N f(x)dx = [\ln x]_1^N = \ln N \to \infty$ as $N \to \infty$, so the series s diverges. To summarise, s converges if and only if $q > 1$.

Further examples

Here is a collection of further examples of convergence of infinite series, using the above methods.

i) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$. Ratio test: We have $a_n = \frac{2^n}{n!}$ so $\frac{a_{n+1}}{a_n} = \frac{2}{n+1} \to 0$ as $n \to \infty$. Thus the series converges. Note: The same thing works with 2 replaced by any other real number, so the series $\sum_{n=1}^{\infty} \frac{a^n}{n!}$ converges for any $a \in \mathbb{R}$.

ii) $\sum_{n=1}^{\infty} \frac{1}{2(n!)+1}$. Quotient test (with the previous one): We have $b_n = \frac{2^n}{n!}$ and $a_n = \frac{1}{2(n!)+1}$, so

$$\frac{a_n}{b_n} = \frac{n!}{2^n \cdot (2(n!)+1)} = \frac{n!}{2(n!)+1}\frac{1}{2^n} < \frac{1}{2^n} \to 0 \text{ as } n \to \infty.$$

Thus the series converges. Notice that as this limit is 0 and thus less than 1, the comparison test would likewise imply converge in this case.