Lecture 6

iii) $\sum_{n=1}^{\infty} r^n \cdot \ln n$, where $r < 1$. Cauchy's root test: $\sqrt[n]{r^n \cdot \ln n} = r \cdot \sqrt[n]{\ln n}$. Now note that for $n \geq 1$, we have $\ln n \leq n$ (for $n = 1$ we have $\ln 1 = 0 < 1$; now take derivative of $n - \ln n$ to show that the function grows). Thus

$$1 \leq \sqrt[n]{\ln n} \leq \sqrt[n]{n},$$

and since $\sqrt[n]{n}$, $n = 1, 2, 3, \ldots$ is a decreasing sequence (compute its derivative!), the sandwiched expression $\sqrt[n]{\ln n}$ tends to 1 as $n \to \infty$.
Therefore, $r \cdot \sqrt[n]{\ln n} \to r$ as $n \to \infty$, and since $r < 1$ the series converges.

Series with negative terms

A series $\sum a_n$ is called *alternating* if $a_n = (-1)^n |a_n|$ or $a_n = (-1)^{n+1} |a_n|$, that is, if the sign alternates every other step, for example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Leibniz test for alternating series

If $s = \sum_{n=1}^{\infty} a_n$ is an alternating series such that $a_n \to 0$ and $|a_{n+1}| < |a_n|$, for all n (i.e., the sequence (a_n) is decreasing), then s converges.

Example 6.1. The "alternating harmonic series"

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \cdots$$

converges (unlike the harmonic series itself), because $|(-1)^{n+1} \frac{1}{n}| = \frac{1}{n} \to 0$ as $n \to \infty$, and $\frac{1}{n+1} < \frac{1}{n}$.

Some terminology (for arbitrary series of real or complex numbers)

Definition 6.2. $\sum a_n$ converges *absolutely* if $\sum |a_n|$ converges. $\sum a_n$ converges *conditionally* if $\sum |a_n|$ diverges but $\sum a_n$ converges.

Example 6.3.

- The "alternating harmonic series" above converges conditionally because it converges even though the harmonic series diverges.
- $\sum \frac{(-1)^{n+1}}{n^2}$ converges absolutely, because $\sum \frac{1}{n^2}$ converges; see (4.3).

Fact. An absolutely convergent series $\sum a_n$ converges.

Fact. If a series is absolutely convergent, we can rearrange the terms in the sum in any way we want, and the series will still converge to the same value.
In conditionally convergent series the order of the terms matters! Moreover, for any $q \in \mathbb{R}$ or $q = \pm\infty$ it is possible to rearrange the terms of a conditionally convergent series such that it will converge to q.

Power series

Let (c_n), $n = 0, 1, 2, \dots$ be a sequence of numbers. A series of the form

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

is called a *power series* (the c_n are called its *coefficients*).

Example 6.4. If $c_n = a$ is a constant, then $P(x) = \sum_{n=0}^{\infty} a x^n = a + ax + ax^2 + \cdots$ is a geometric series. Recall that this converges if $|x| < 1$. For every $x \in \mathbb{R}$ such that $|x| < 1$, we have a series, so we get a function

$$P : (-1, 1) \longrightarrow \mathbb{R}, \qquad P(x) = \sum_{n=0}^{\infty} c_n x^n.$$

The partial sums $s_N = \sum_{n=0}^{N} c_n x^n$ approximate the limit $s = \sum_{n=0}^{\infty} c_n x^n$, so the function P (which is not in general a polynomial) can be approximated by polynomials, as long as $x \in (-1, 1)$, that is, locally in a neighbourhood of 0.

The above example is fundamental in mathematics. The fact that we can approximate many (complicated) functions by polynomials locally around a point is the basis for much of analysis and numerical computations in science.

Example 6.5. Recall from (3.4) that for x such that $|x| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1 - x).$$

So, $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = P(x)$ and the partial sums are

$$P_N(x) = x + \cdots + \frac{x^N}{N}.$$

Now, $P(x) - P_N(x) = \frac{x^{N+1}}{N+1} + \frac{x^{N+2}}{N+2} + \cdots$, so

$$\frac{P(x) - P_N(x)}{x_{N+1}} = \frac{1}{N+1} + \frac{x}{N+2} + \cdots \longrightarrow \frac{1}{N+1} \quad \text{as } x \longrightarrow 0.$$

Thus, as $x \to 0$, the ratio $\frac{P(x)-P_N(x)}{x_{N+1}}$ is close to $\frac{1}{N+1}$, and therefore

$$P(x) - P_N(x) \text{ is approximated by } \frac{x^{N+1}}{N+1}.$$

We write this

$$P(x) - P_N(x) \sim \frac{x^{N+1}}{N+1}.$$

The difference $P(x) - P_N(x)$ should be thought of as the error when we approximate the function $P(x)$ by the partial sum $P_N(x)$. Clearly, increasing N, we can make this error as small as we want.