# Lecture 7

### Convergence of power series

We use the ratio test for $\sum_{n=0}^{\infty} |c_n x^n|$, that is, we look at absolute convergence:

$$p = \lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = \lim_{n\to\infty} |x| \cdot \left| \frac{c_{n+1}}{c_n} \right|.$$

We want to know the values of $x$ for which the series converges.

- If $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = 0$, then $p = 0$ for all $x$, so it converges.
- If $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = c \in \mathbb{R}$, then $p = |x|c$, so $p < 1$ iff $|x| < \frac{1}{c}$. Thus it converges if $|x| < \frac{1}{c}$ and diverges if $|x| > \frac{1}{c}$. If $|x| = \frac{1}{c}$ anything can happen, so we need to check convergence separately.

**Example 7.1.** $P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$. We have $c_n = \frac{1}{n!}$, so

$$\frac{c_{n+1}}{c_n} = \frac{1}{n+1} \longrightarrow 0.$$

Thus the series converges for all $x$. In fact, it converges to $e^x$.

**Example 7.2.** $P(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = -\ln(1-x)$ by (3.4). We have $c_n = \frac{1}{n}$, so

$$\frac{c_{n+1}}{c_n} = \frac{n}{n+1} \longrightarrow 1.$$

Thus the series converges if $|x| < 1$ and diverges if $|x| > 1$. For $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{1}{n+1}$, which is just the harmonic series (diverges!). If $x = -1$, we have the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$, which converges by (6.1) (it's the "alternating harmonic series").

Thus our series converges precisely for $x$ in the half-open interval $[-1, 1)$.

**Definition 7.3.** The set of $x \in \mathbb{R}$ such that $P(x)$ converges is called the *interval of convergence* of the power series. This interval has *radius* $R = \frac{1}{c} = \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|}$ (i.e., half the length of the interval).

*Remark.* We can also use the root test to find the radius of convergence. Namely, $R = \frac{1}{d} = \frac{1}{\lim_{n\to\infty} |c_n|^{1/n}}$ (if the limit exists).

*Remark.* We may also be able to compute the radius (and interval) of convergence if the limits above do not exist.

**Example 7.4.** $P(x) = \sum_{n=0}^{\infty} 2^n x^{2n}$ has all odd coefficients equal to zero, so the ratio test cannot be applied. As $\lim_{n\to\infty} |c_{2n}|^{1/2n} = \sqrt{2} \neq 0$, the root test cannot be applied either.

However, we can write $y = x^2$, and then the series becomes $P(x) = \sum_{n=0}^{\infty} 2^n x^{2n} = \sum_{n=0}^{\infty} 2^n y^n = \sum_{n=0}^{\infty} a_n y^n$, so applying the root test we obtain that the series converges if $|y| < \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|} = \frac{1}{2}$ and diverges if $|y| > 1/2$. Since $y = x^2$, we see that $P(x)$ converges if $|x| < 1/\sqrt{2}$ and diverges if $|x| \geq 1/\sqrt{2}$, so the interval of convergence is $(-1/\sqrt{2}, 1/\sqrt{2})$.

*Remark.* In the case of two-variable analysis (or complex analysis), the convergence will be in a circular disc instead of an interval. This is where the terminology radius of convergence comes from.

For example, $P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ has interval of convergence $(-\infty, \infty) = \mathbb{R}$ and the radius is $\infty$. The series $P(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$ has interval of convergence $[-1, 1)$ and radius $R = 1$.

**Series with complex numbers**

Assume that we have a power series $P(z) = \sum_{n=0}^{\infty} c_n z^n$, where $z \in \mathbb{C}$ and $c_n \in \mathbb{C}$. We can consider the real series $\sum |c_n z^n|$, which we know converges if

$$|z| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| < 1.$$

Thus, $\sum c_n z^n$ converges aboslutely if $|z| < \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|}$.

**Example 7.5.** $P(z) = \sum_{n=0}^{\infty} z^n$. We have $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = 1$, so the series converges absolutely for $|z| < 1$ and diverges for $|z| > 1$. For $|z| = 1$ it also diverges, because then $z^n$ does NOT tend to 0 (for $z^n$ to tend to 0 its modulus, which is the distance to the origin, would have to tend to 0).

Note that the set of $z \in \mathbb{C}$ such that $|z| < 1$, is a disc (with radius 1), rather than an interval.

**Operations with power series**

Suppose $P(z)$ and $Q(z)$ are two power series which converge in some disc $|z| < R$. Then:

- $P(z) + Q(z)$ and $aP(z)$ converge, for any $a \in \mathbb{C}$.
- Inside the disc of convergence, we can integrate and differentiate series.

Why can we differentiate series inside the interval/disc of convergence? Well, if $P(z) = \sum c_n z^n$, we know that the radius is

$$R = \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|}.$$

Now, $\frac{dP(z)}{dz} = \sum n c_n z^{n-1}$ and the radius of this series is

$$\frac{1}{\lim_{n\to\infty} \left| \frac{(n+1)c_{n+1}}{n c_n} \right|} = \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|} = R,$$

so the radius of convergence is the *same* as before. This means that taking the derivative does not "destroy" the disc of convergence.

**Example 7.6.** $P(x) = \sum x^n = \frac{1}{1-x}$ for $|x| < 1$. Thus,

$$\begin{aligned} \frac{dP(x)}{dx} &= (x-1)^{-2} = (1 + x + x^2 + \cdots)' = 1 + 2x + 3x^2 + \cdots \\ &= \sum_{n=1}^{\infty} n x^{n-1} = \sum_{k=0}^{\infty} (k+1) x^k. \end{aligned}$$

This series is called the *expansion* of $(x-1)^{-2} = \frac{1}{(x-1)^2}$ (at $x = 0$). In the following, we will now see what this means in general.