Lecture 8

Taylor series

Many (most?) physical phenomena cannot be completely described by polynomial functions (because these are too simple to do the job, in general). However, if we study the behaviour of functions in a small region around a point at a time, then we can approximate the function well by polynomials. This has big advantages from a computational point of view, because to compute polynomials we only need the four standard operations (no square roots, logarithms etc are needed).

Goal: To approximate a function by polynomials which are partial sums of a power series (in some interval around a point).

Recall that if we have a function $y = f(x)$, we can plot its graph in the $x-y$-plane. We want to approximate the graph by something simple. The simplest graph is a line. We can then approximate the graph of f *at a point* a by taking the tangent at a. The slope of this tangent line is of course the derivative $f'(x)$, and the equation for the line is

$$y = f'(a)(x-a) + f(a).$$

This is a first approximation of f and it is called the *first Taylor polynomial* $p_1(x)$.

Why is it a "good" approximation? Well, the error is

$$f(x) - p_1(x) = f(x) - f(a) - f'(a)(x-a) = (x-a)\underbrace{\left(\frac{f(x)-f(a)}{x-a} - f'(a)\right)}_{\to 0 \text{ as } x\to a}.$$

So, as we approach the point $x = a$, the error goes to 0.

We now want to find better approximations, so-called "higher" Taylor polynomials $p_2(x), p_3(x), \ldots$ such that for each n

$$\frac{f(x) - p_n(x)}{(x-a)^n} \longrightarrow 0 \text{ as } x \longrightarrow a.$$

Let x be approximately equal to a. We write this $x \approx a$. We can write

$$\int_a^x f'(t)dt = f(x) - f(a),$$

and so

$$f(x) = f(a) + \int_a^x f'(t)dt. \tag{☺}$$

Apply the equality above to the function f' at point $t \in (a, x)$ (or (x, a)):

$$f'(t) = f'(a) + \int_a^t f''(s)ds \approx f'(a) + (t-a)f''(a).$$

Plug this into ☺ :

$$\begin{aligned} f(x) &\approx f(a) + \int_a^x \big(f'(a) + (t-a)f''(a)\big)\,dt \\ &= f(a) + f'(a)(x-a) + f''(a)\int_a^x (t-a)dt \\ &= f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} = p_2(x). \end{aligned}$$

This is our second Taylor polynomial $p_2(x)$.

Definition 8.1. In general, we get the Nth *Taylor polynomial*:

$$\begin{aligned} p_N(x) &= f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \frac{(x-a)^3}{3!}f^{(3)}(a) \\ &\quad + \cdots + \frac{(x-a)^N}{N!}f^{(N)}(a) \\ &= \sum_{n=0}^{N} \frac{(x-a)^n}{n!}f^{(n)}(a). \end{aligned}$$

(note that $0! = 1$).

Example 8.2. $f(x) = x^2 + 2x + 1$ and $a = 0$. Then

$$\begin{aligned} f(a) &= 1 & & \\ f'(a) &= 2 & p_1(x) &= 1 + 2x \\ f''(a) &= 2 & p_2(x) &= 1 + 2x + \frac{2x^2}{2} = f(x) \\ f^{(3)}(a) &= 0 & p_3(x) &= p_2(x) \\ \cdots &= 0 & &\cdots \\ \cdots &= 0 & p_N(x) &= p_2(x). \end{aligned}$$

In general, the Nth Taylor polynomials of a polynomial function $f(x)$ of degree n will eventually (for $N \geq n$) be the polynomial $f(x)$ itself.

Here are some other examples of Taylor polynomials:

Example 8.3. Let $f(x) = \sin x$. Compute $p_5(x)$, around $a = 0$.

	Values at $a = 0$
$f(x) = \sin x$	0
$f'(x) = \cos x$	1
$f''(x) = -\sin x$	0
$f^{(3)}(x) = -\cos x$	-1
$f^{(4)}(x) = \sin x$	0
$f^{(5)}(x) = \cos x$	$1.$

Thus $p_5(x) = 0 + x + \frac{x^2}{2} \cdot 0 + \frac{x^3}{3!}(-1) + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

Remark: Instead of a Taylor series "around" a point a, some authors say "centered at a", "about a", or simply "at a". The point is (no pun intended) that the Taylor expansion is only valid in sufficiently small neighbourhoods around the point a.

Example 8.4. Let $f(x) = \ln x$. Compute $p_4(x)$ around $a = 1$.

	Values at $a = 1$
$f(x) = \ln x$	0
$f'(x) = \frac{1}{x}$	1
$f''(x) = -\frac{1}{x^2}$	-1
$f^{(3)}(x) = \frac{2}{x^3}$	2
$f^{(4)}(x) = \frac{-3!}{x^4}$	$-3!$

Thus $p_4(x) = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$.

Setting $y = x - 1$, we get the Taylor polynomial $p_4(y)$ for $f(y) = \ln(1 + y)$ around $a = 0$:

$$y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4}.$$