Lecture 9

Definition 9.1. The *Taylor series* of $f(x)$ around a is a power series

$$\sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}.$$

Each partial sum is a Taylor polynomial. (For $a = 0$ this is also called a Maclaurin series.)

Note that the Taylor series is defined only if $f(x)$ is infinitely differentiable at $x = a$ (i.e., if $f^{(n)}(a)$ exists for all $n \geq 1$). If the Taylor series converges, it will equal the value of $f(x)$ for x close to a. Further away from a it may not approximate $f(x)$ well. For some nice functions, the Taylor series equals $f(x)$ everywhere:

Example 9.2. The Taylor series of e^x and $\sin x$ around $a = 0$ are

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

Note that these converge absolutely for all $x \in \mathbb{R}$.

By contrast, the Taylor series for $\ln(1+x)$ does not converge everywhere:

Example 9.3. In (8.4) we see a recurring pattern in the partial sums, so

$$\ln(1+y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} y^n$$

(we could use x instead of y here). For convergence,

$$\left| \frac{\frac{(-1)^{n+2}}{(n+1)} y^{n+1}}{\frac{(-1)^{n+1}}{n} y^n} \right| = |y| \cdot \frac{n}{(n+1)} \longrightarrow |y|,$$

so it converges if $|y| < 1$. For $|y| > 1$ it diverges and is therefore *not* a good approximation.

Conclusion. A Taylor series (or any power series) converges absolutely in a disc/interval about its expansion point a, and diverges outside of it.

Theorem 9.4 (Taylor's theorem)**.** *Suppose f is $N+1$ times differentiable in a neighborhood of a, x belongs to this neighborhood, and $f^{(N+1)}$ is continuous between a and x. Then*

$$f(x) = p_N(x) + \underbrace{\frac{f^{(N+1)}(t)}{(N+1)!}(x-a)^{N+1}}_{=R_N(x)},$$

for some t such that $a < t < x$ (or $x < t < a$). Here $p_N(x)$ is the Nth Taylor polynomial and $R_N(x)$ is the remainder/error in Lagrange form.

By Taylor's theorem, we can bound the error when $f(x)$ is approximated by $p_N(x)$:

Example 9.5. Compute $\cos(0.2)$ with error less than 10^{-4}. Plan: Let $f(x) = \cos x$ and $p_N(x)$ be the Taylor polynomial around $a = 0$. Find N such that

$$|\cos(x) - p_N(x)| < 10^{-4},$$

and then compute $p_N(0.2)$.

The error term in Lagrange form is

$$R_N(x) = \frac{f^{(N+1)}(t)}{(N+1)!} x^{N+1}.$$

The derivatives $f^{(n)}(x)$ of $f(x) = \cos x$ are all either $\pm \sin x$ or $\pm \cos x$. Thus, $|f^{(N+1)}(t)| \leq 1$, and so

$$|R_N(x)| \leq \frac{|x|^{N+1}}{(N+1)!}.$$

This implies (by Taylor's theorem) that

$$|\cos x - p_N(x)| \leq \frac{|x|^{N+1}}{(N+1)!}$$

and hence, setting $x = 0.2$,

$$|\cos(0.2) - p_N(0.2)| \leq \frac{|0.2|^{N+1}}{(N+1)!}.$$

Now find N such that $\frac{|0.2|^{N+1}}{(N+1)!} \leq 10^{-4}$. Try $N = 3$:

$$\frac{(2 \cdot 10^{-1})^{3+1}}{(3+1)!} = 16 \cdot 10^{-4}/24 < 10^{-4},$$

so this will work! We just need to evaluate $p_3(0.2)$:

$$p_3(x) = 1 + x(-\sin 0) + \frac{x^2}{2}(-\cos 0) + \frac{x^3}{3!}(\sin 0) = 1 - \frac{x^2}{2},$$

so

$$p_3(0.2) = 1 - 0.02 = 0.98.$$

Thus,

$$\cos(0.2) \approx 0.98,$$

and this value has error at most 10^{-4}, that is, it is correct to at least four decimal places.

Check with a calculator (using radians, not degrees, for angles): $\cos(0.2) = 0.980066\ldots$