# Single Maths A – Epiphany 2018

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All the times and room numbers are the same as last term.

Homeworks will be set on Fridays, during the lecture. Hand in your homeworks in the lockers in CM117. The deadline is strictly 5pm, before the start of the lecture the following week. Tutorials start in week 12.

All the course material will be made available on DUO and on the class page
http://www.maths.dur.ac.uk/users/pavel.tumarkin/SMA.

WARNING! The lectures will *approximately* correspond to the notes below. However things may change and there will be more material at the end hopefully. The content of the course is defined by the actual lectures, not by the following notes!

### Short outline of this term's content

- Series, Taylor series
- Matrices, systems of linear equations
- Vector spaces, linear maps, eigenvalues, eigenvectors
- Groups

Textbook: Same as last term (Riley et al).

## Series

Here is a problem about series which is part of a popular story[1]:

> Two trains are 20 miles apart on the same track heading towards each other at 10 mi/h, on a collision course. At the same time, a fly takes off from the nose of one train at 20 mi/h, towards the other train. As soon as the fly reaches the other train, it turns around and heads off at 20 mi/h back towards the first train. It continues to do this until the trains collide.
> Question: How far does the fly fly before the collision?

There is a relatively easy solution: The trains will collide after exactly 1 hour (since they will each have gone 10 miles by then). Since the speed of the fly is 20 mi/h, it will have flown 20 miles by the time the trains collide. (This is sometimes described as the physicists solution).

Another solution goes as follows: The fly is twice as fast as the trains, so on the first leg of its flight, it will cover $x_1$ miles, while the train going in the opposite direction will have covered $x_1/2$ miles. The total, $x_1 + \frac{x_1}{2}$ must equal the initial distance: 20 miles. Thus

$$3x_1/2 = 20 \Longrightarrow x_1 = \frac{2}{3}20.$$

For the second leg of the flight, the new distance between the trains is $20 - 2\frac{x_1}{2} = 20 - \frac{2}{3}20 = \frac{1}{3}20$ miles, so by the same argument the fly will cover

$$x_2 = \frac{2}{3}(\frac{1}{3}20) \text{ miles.}$$

Similarly, the distance of the third leg is

$$x_3 = \frac{2}{3}(\frac{1}{3}20 - 2\frac{x_2}{2}) = \frac{2}{3}\frac{1}{3}(\frac{1}{3}20).$$

In general, the $n$th distance the fly covers is

$$x_n = \frac{2}{3}20(\frac{1}{3})^{n-1}.$$

Continuing this way infinitely many steps, and summing all the distances, we get the total distance the fly covers:

$$\begin{aligned} x_1 + x_2 + \cdots = \frac{2}{3}20 + \frac{2}{3}20\frac{1}{3} + \frac{2}{3}20(\frac{1}{3})^2 + \cdots \\ = \frac{2}{3}20\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots\right). \end{aligned}$$

We now need to evaluate the infinite series in the brackets. Let

$$x = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots.$$

[1] The story involves the mathematician John von Neumann. It is not examinable.

Then

$$3(x-1) = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots = x.$$

Solving for $x$, we get $x = 3/2$, so the solution to the original problem is $\frac{2}{3}20 \cdot x = 20$ miles, which luckily agrees with the first solution!

It is said that physicists instictively solve this problem in the first way, while mathematicians try to solve it using an infinite series. This may just be a myth, but it begs the question: Why do we even consider the more complicated solution? The answer is that many problems will not have an easy solution, so that the techniques of infinite series *has* to be used, and there is no alternative. That's why we study them.

## Terminology

An individual piece of a series is called a *term*, often denoted by a subscript. For example, in the series

$$x_1 + x_2 + \cdots,$$

each $x_i$ is a term, for $i = 1, 2, \ldots$.

There are *finite series*

$$x_1 + x_2 + \cdots + x_N = \sum_{i=1}^{N} x_i,$$

and *infinite series* (with infinitely many terms)

$$x_1 + x_2 + \cdots = \sum_{i=1}^{\infty} x_i.$$

**Example 2.1.** In the fly problem, we had the infinite series

$$1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots = \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i$$

(note that $i$ starts from 0). The *sum* of this series is its value, which we computed to be $3/2$.

If we write this series as $\sum_{i=1}^{\infty} x_i$, with $x_i = \left(\frac{1}{3}\right)^i$, then we have a *recurrence relation* $x_{i+1} = \frac{1}{3}x_i$, for each $i \geq 0$. A recurrence relation is a rule for computing a term from some of the previous terms.

Our goals are:

- To understand how to compute finite series $\sum_{i=1}^{N} x_i$.
- To define and investigate infinite series. Does the sum exist? If so, how do we compute it?

### Series basics

**Definition 2.2.** Let $s = \sum_{i=1}^{\infty} a_i$ be an infinite series. A *partial sum* is

$$a_1 + a_2 + \cdots + a_N = \sum_{i=1}^{N} a_i = s_N.$$

In this way, we get a *sequence* of partials sums: $s_1, s_2, \ldots$. A series can also start at $i = 0$ or any other integer, even negative ones.

The series $s = \sum_{i=1}^{\infty} a_i$ *converges* if the sequence $s_1, s_2$ has a limit, that is if $\lim_{N\to\infty} s_n$ exists. This limit is then denoted by $s$, and it is the sum of the series. If the series does not converge, it is said to *diverge*.

**Example 2.3.**

- $1+0+0+0+\cdots$ converges. The sequence of partial sums $s_1, s_2, \ldots$ is $1, 1, 1, \ldots$ whose limit is just 1.
- $1+1+1+\cdots$ diverges. The sequence of partial sums $s_1, s_2, \ldots$ is $1, 2, 3, \ldots$, which does not have a limit.

**Example 2.4.** Arithmetic series (finite). This is where the difference between two consecutive terms is constant (e.g., $0 + 2 + 4 + 6 + \cdots$ has constant difference 2). Therefore, we can write an arithmetic series as

$$(a + d) + (a + 2d) + \cdots + (a + Nd) = s_N = \sum_{i=1}^{N} a_i, \text{ where } a_i = a + id.$$

where $d$ is the difference. How do we compute these? Write the series once and once in reverse order:

$$\begin{aligned} s_N &= (a + d) + (a + 2d) + \cdots + (a + Nd) \\ s_N &= (a + Nd) + (a + (N-1)d) + \cdots + (a + d). \end{aligned}$$

The sum of two terms lying above each other is always $2a + (N + 1)d$, so

$$\begin{aligned} 2s_N &= \underbrace{(a + d) + (a + Nd) + \cdots + (a + Nd) + (a + d)}_{N \text{ pairs of terms}} \\ &= N(2a + (N + 1)d). \end{aligned}$$

Thus $s_N = \frac{N}{2}(2a + (N + 1)d)$. If $a = 0$ and $d = 1$, we get

$$1 + 2 + \cdots + N = \frac{N(N + 1)}{2}.$$

**Example 2.5.** Geometric series. Here the quotient of two consecutive terms is constant (e.g., $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ with constant quotient 1/2). They look like

$$s_N = \sum_{i=0}^{N} ar^i = a + ar + ar^2 + \cdots + ar^N$$

and we will always assume that $r \neq 1$ (if $r = 1$ it is an arithmetic series with difference 0). How do we compute these? If we multiply the series by $r$ and add $a$ we get

$$rs_N + a = a + r(a + ar + \cdots + ar^N) = a + ar + ar^2 + \cdots + ar^{N+1} = s_N + ar^{N+1}.$$

Solving for $s_N$, we get

$$s_N = a\frac{r^{N+1} - 1}{r - 1}.$$

If instead we consider the infinite series

$$\sum_{i=0}^{\infty} ar^i,$$

then have just seen that the partial sums are $s_N$ as above. Now,

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} a\frac{r^{N+1} - 1}{r - 1} = a\frac{1}{1 - r} + \lim_{N\to\infty} a\frac{r^{N+1}}{r - 1} = \begin{cases} 0 & \text{if } a = 0, \\ a\frac{1}{1-r} & \text{if } |r| < 1. \end{cases}$$

This is because if $|r| < 1$, then

$$\lim_{N\to\infty} a\frac{r^{N+1}}{r - 1} = \frac{a}{r - 1} \lim_{N\to\infty} r^{N+1} = 0.$$

If $|r| \geq 1, a \neq 0$, then there is no limit. In particular, we have

$$1 + r + r^2 + \cdots = \frac{1}{1 - r}$$

whenever $|r| < 1$. We will use this important identity later.

# Lecture 3

## The difference method

This is a method which is sometimes useful to calculate finite or infinite series.

**Example 3.1.** Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Then the partial sums are $s_N = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{N(N+1)}$. Observe that

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

(partial fraction decomposition). Using this, we can write

$$s_N = \frac{1}{1} \underbrace{-\frac{1}{2} + \frac{1}{2}}_{=0} - \frac{1}{3} + \cdots + \frac{1}{N-1} \underbrace{-\frac{1}{N} + \frac{1}{N}}_{=0} - \frac{1}{N+1} = 1 - \frac{1}{N+1} = \frac{N}{N+1}.$$

In particular, letting $N \to \infty$, we get the sum of the infinite series: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Whenever we have a series $\sum_{n=1}^{\infty} a_n$ where we can write

$$a_n = f(n) - f(n+1)$$

for some function $f$ (in the above, we had $f(n) = \frac{1}{n}$), then

$$s_N = f(1) - f(N+1)$$

and we can compute the sum of the infinite series as $\lim_{N\to\infty} f(1) - f(N+1)$ which is equal to $f(1) - \lim_{N\to\infty} f(N+1)$. This is called the *difference method* of computing the sum.

Here is an extension of the method when we jump more than one step each time:

**Example 3.2.** Let $s = \sum_{n=1}^{\infty} \frac{1}{n(n+3)}$. We can write

$$\frac{1}{n(n+3)} = \frac{1}{3}(\frac{1}{n} - \frac{1}{n+3}) = f(n) - f(n+3),$$

where $f(n) = \frac{1}{3n}$. Thus

$$\begin{aligned} s_N &= f(1) - f(4) + f(2) - f(5) + f(3) - f(6) + f(4) - f(7) + \cdots + \\ &\quad + f(N-2) - f(N+1) + f(N-1) - f(N+2) + f(N) - f(N+3) \\ &= f(1) + f(2) + f(3) - f(N+1) - f(N+2) - f(N+3). \end{aligned}$$

Letting $N \to \infty$, we is we get

$$s = \lim_{N\to\infty} s_N = f(1) + f(2) + f(3) = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18},$$

since $f(N) \to 0$ as $N \to \infty$ (so $f(N+1)$ etc vanish in the limit).

Here is another example using the difference method:

**Example 3.3.** Compute $s_N = \sum_{n=1}^{N} n^3$. We need to find a function $f$ such that $n^3 = f(n) - f(n+1)$. First, note that if $f(n) = (n(n-1))^2$, then

$$f(n) - f(n+1) = -4n^3,$$

which isn't quite right. But if we modify this to cancel the $-4$ factor and take $g(n) = -\frac{1}{4}(n(n-1))^2$, we do get

$$g(n) - g(n+1) = n^3.$$

Then the difference method now says that $s_N = g(1) - g(N+1) = -\frac{1}{4}(0 - (N(N+1))^2) = \frac{(N(N+1))^2}{4}$.

## Manipulating series

In order to simplify series and to be able to sum them it is sometimes useful to transform a series into another one, for example by differentiating or integrating it. In order to manipulate infinite series, we always need to ensure that the series converge.

**Example 3.4.** Let $s(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \cdots$. Differentiating, we get

$$\frac{ds}{dx} = \sum_{n=0}^{\infty} \left( \frac{x^{n+1}}{n+1} \right)' = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ if } |x| < 1 \text{ by (2.5).}$$

Integrating, we get $s(x) = \int \frac{1}{1-x} dx = -\ln(1-x) + c$. To determine the constant $c$, note that $s(0) = 0$, so we must have $c = 0$ and

$$s(x) = -\ln(1-x) \text{ for } |x| < 1.$$

**Warning:** Here is an example of how things can go wrong if the series does not converge. Let

$$s = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + \cdots .$$

If we multiply both sides by 2, we get

$$2s = 2 + 4 + 8 + \cdots = s - 1 \Longrightarrow s = -1,$$

which is clearly nonsense since we are summing positive numbers. Moral of the story: Do not manipulate diverging series. It is therefore important to know when a series converges, and this is what we will look at now.

# Lecture 4

## Convergence, necessary conditions, tests

**Proposition 3.5** (Necessary condition for convergence). *If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.*

In other words, if $a_n$ does not go to zero, then the series diverges. This immediately tells us that the series $2+4+8+\cdots$ diverges, and similarly $\sum_{n=0}^{\infty}(-1)^n = 1-1+1-1+\cdots$ diverges. However, the converse of this proposition is not true as the following example shows.

**Example 4.1.** (The harmonic series). The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series. We will show that it diverges. Some of the first partial sums are

$$s_1 = 1, \quad s_2 = 1 + \frac{1}{2} = \frac{3}{2}, \quad s_4 = s_2 + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2.$$

We can therefore guess that we always have

$$s_{2^n} \geq \frac{n}{2} + 1.$$

We now prove this by induction. Assume it is true for some $n \geq 1$. Now

$$\begin{aligned}
s_{2^{n+1}} &= s_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n} \\
&> s_{2^n} + \frac{1}{2^n+2^n} + \frac{1}{2^n+2^n} + \cdots + \frac{1}{2^n+2^n} \\
&= s_{2^n} + \frac{2^n}{2^{n+1}} \geq \frac{n}{2} + 1 + \frac{1}{2} \quad \text{(by the induction hypothesis)} \\
&= \frac{n+1}{2} + 1.
\end{aligned}$$

This shows that $s_{2^n} \geq \frac{n}{2} + 1$, for all $n \geq 1$, and thus the partial sums tend to $\infty$ as $n \to 0$.

Assume now that all the terms in the series are $\geq 0$. We have the following convergence tests:

### Comparison test

Assume that $S = \sum_{n=0}^{\infty} b_n$ converges and $a_n \leq b_n$ for all $n$ large enough. Then $s = \sum_{n=0}^{\infty} a_n$ converges. Reason: $s_N \leq S_N$ for each $N$.

Equivalently, if $s = \sum_{n=0}^{\infty} a_n$ diverges, and $a_n \leq b_n$, then $S = \sum_{n=0}^{\infty} b_n$ diverges.

**Example 4.2.** Let $s = \sum_{n=0}^{\infty} \frac{2^n+3^n}{4^n}$. Then

$$a_n = \frac{2^n+3^n}{4^n} \leq \frac{3^n+3^n}{4^n} = 2\left(\frac{3}{4}\right)^n = b_n.$$

But $\sum_{n=0}^{\infty} b_n = 2\sum_n \left(\frac{3}{4}\right)^n = \frac{2}{1-3/4}$ (geometric series with $|x| < 1$, see (2.5)).

**Quotient test**

Let $s = \sum_{n=0}^{\infty} a_n$ and $S = \sum_{n=0}^{\infty} b_n$ as before. Compute $p = \lim_{n\to\infty} \frac{a_n}{b_n}$. Then:

- if $p = 0$ and $S$ converges, then $s$ converges.
- if $p = \infty$ and $s$ converges, then $S$ converges.
- If $0 < p < \infty$, then $s$ converges if and only if $S$ does.
- If $p$ doesn't exist (i.e., $\frac{a_n}{b_n}$ does not have a limit), then we can't say anything.

**Example 4.3.** Let $s = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_n a_n$ and $S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_n b_n$. Now

$$p = \lim \frac{a_n}{b_n} = \lim \frac{1/n^2}{\frac{1}{n(n+1)}} = \lim \frac{n+1}{n} = 1.$$

Thus, by the quotient test, $s$ converges if and only if $S$ does. But we know that $S$ converges by (3.1), so we conclude that $s$ also converges.

We can also use the comparison test to show that $s$ converges: We have

$$s = \sum_{n=1}^{\infty} \frac{1}{n^2} = s = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \leq 1 + \sum_{n=0}^{\infty} \frac{1}{n(n+1)},$$

where we know that the last series converges, by (3.1). Thus the comparison test implies convergence also for $s$.

Moreover, $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges for any $k \geq 2$ because $\lim \frac{1/n^k}{1/n^2} = \lim \frac{1}{n^{k-2}} = 0$, for any $k > 2$.

As we saw in the above example, it is often possible to use several different tests to show convergence (but sometimes only one test works; it depends on the series).

# Lecture 5

### Ratio test

Let $s = \sum_{n=0}^{\infty} a_n$ and $p = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, if the limit exists. Then

- if $p < 1$, then $s$ converges.
- if $p > 1$, then $s$ diverges.
- if $p = 1$, then anything can happen ($s$ may converge, may diverge).

**Example 5.1.** Let $s = \sum_{n=0}^{\infty} \frac{n^2}{2^n} = \sum a_n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{1}{2}\frac{(n+1)^2}{n^2} = \frac{1}{2}(1+\frac{1}{n})^2 \to \frac{1}{2} \text{ as } n \to \infty.$$

Since $\frac{1}{2} < 1$, we conclude that $s$ converges.

### Cauchy's root test

Let $s = \sum_{n=0}^{\infty} a_n$ and $p = \lim_{n\to\infty} \sqrt[n]{a_n}$, if the limit exists. Then

- if $p < 1$, then $s$ converges.
- if $p > 1$, then $s$ diverges.
- if $p = 1$ anything can happen.

**Example 5.2.** Does $s = \sum_{n=1}^{\infty} \frac{2n}{3^n}$ converge or diverge? The $n$th root of the $n$th term is

$$\left(\frac{2n}{3^n}\right)^{1/n} = \frac{(2n)^{1/n}}{3},$$

so we need to compute the limit of $\frac{(2n)^{1/n}}{3}$ as $n \to \infty$. Taking the logarithm of the numerator, we have

$$\ln((2n)^{1/n}) = \frac{\ln(2n)}{n},$$

and by l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\ln(2n)}{n} = \lim_{n\to\infty} \frac{2/2n}{1} = 0.$$

Exponentiating both sides, we get $\lim((2n)^{1/n}) = e^0 = 1$, so

$$\lim \frac{(2n)^{1/n}}{3} = \frac{1}{3} < 1.$$

Thus, by the root test, $s$ converges.

### Integral test

Suppose we can find a *decreasing* function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(n) = a_n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\lim_{N\to\infty} \int_1^N f(x)dx \quad \text{exists.}$$

The idea here is that the integral is a good approximation of the series, so one converges iff the other does.

**Example 5.3.** Let $s = \sum_{n=1}^{\infty} \frac{1}{n^q}$, $q > 0$. Then we can take $f(x) = \frac{1}{x^q}$, which is decreasing. We have, if $q \neq 1$:

$$\int_1^N f(x)dx = \left[\frac{1}{-q+1} x^{-q+1}\right]_1^N = \frac{1}{1-q}(N^{-q+1} - 1).$$

Thus:

- If $q > 1$, we get $\lim_{N\to\infty} \frac{1}{N^{q-1}} = 0$, so $\lim \int_1^N f(x)dx = \frac{1}{q-1}$, hence the series $s$ converges.
- If $q < 1$, then $\lim_{N\to\infty} \frac{1}{N^{q-1}} = \infty$, so the integral has no limit, hence the series $s$ diverges.

If $q = 1$, we have $f(x) = \frac{1}{x}$, so $\int_1^N f(x)dx = [\ln x]_1^N = \ln N \to \infty$ as $N \to \infty$, so the series $s$ diverges. To summarise, $s$ converges if and only if $q > 1$.

### Further examples

Here is a collection of further examples of convergence of infinite series, using the above methods.

*i)* $\sum_{n=1}^{\infty} \frac{2^n}{n!}$. Ratio test: We have $a_n = \frac{2^n}{n!}$ so $\frac{a_{n+1}}{a_n} = \frac{2}{n+1} \to 0$ as $n \to \infty$. Thus the series converges. Note: The same thing works with 2 replaced by any other real number, so the series $\sum_{n=1}^{\infty} \frac{a^n}{n!}$ converges for any $a \in \mathbb{R}$.

*ii)* $\sum_{n=1}^{\infty} \frac{1}{2(n!)+1}$. Quotient test (with the previous one): We have $b_n = \frac{2^n}{n!}$ and $a_n = \frac{1}{2(n!)+1}$, so

$$\frac{a_n}{b_n} = \frac{n!}{2^n \cdot (2(n!)+1)} = \frac{n!}{2(n!)+1}\frac{1}{2^n} < \frac{1}{2^n} \to 0 \text{ as } n \to \infty.$$

Thus the series converges. Notice that as this limit is 0 and thus less than 1, the comparison test would likewise imply converge in this case.

# Lecture 6

*iii)* $\sum_{n=1}^{\infty} r^n \cdot \ln n$, where $r < 1$. Cauchy's root test: $\sqrt[n]{r^n \cdot \ln n} = r \cdot \sqrt[n]{\ln n}$. Now note that for $n \geq 1$, we have $\ln n \leq n$ (for $n = 1$ we have $\ln 1 = 0 < 1$; now take derivative of $n - \ln n$ to show that the function grows). Thus

$$1 \leq \sqrt[n]{\ln n} \leq \sqrt[n]{n},$$

and since $\sqrt[n]{n}$, $n = 1, 2, 3, \ldots$ is a decreasing sequence (compute its derivative!), the sandwiched expression $\sqrt[n]{\ln n}$ tends to 1 as $n \to \infty$.
Therefore, $r \cdot \sqrt[n]{\ln n} \to r$ as $n \to \infty$, and since $r < 1$ the series converges.

## Series with negative terms

A series $\sum a_n$ is called *alternating* if $a_n = (-1)^n |a_n|$ or $a_n = (-1)^{n+1}|a_n|$, that is, if the sign alternates every other step, for example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

### Leibniz test for alternating series

If $s = \sum_{n=1}^{\infty} a_n$ is an alternating series such that $a_n \to 0$ and $|a_{n+1}| < |a_n|$, for all $n$ (i.e., the sequence $(a_n)$ is decreasing), then $s$ converges.

**Example 6.1.** The "alternating harmonic series"

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \cdots$$

converges (unlike the harmonic series itself), because $|(-1)^{n+1}\frac{1}{n}| = \frac{1}{n} \to 0$ as $n \to \infty$, and $\frac{1}{n+1} < \frac{1}{n}$.

### Some terminology (for arbitrary series of real or complex numbers)

**Definition 6.2.** $\sum a_n$ converges *absolutely* if $\sum |a_n|$ converges. $\sum a_n$ converges *conditionally* if $\sum |a_n|$ diverges but $\sum a_n$ converges.

**Example 6.3.**

- The "alternating harmonic series" above converges conditionally because it converges even though the harmonic series diverges.
- $\sum \frac{(-1)^{n+1}}{n^2}$ converges absolutely, because $\sum \frac{1}{n^2}$ converges; see (4.3).

**Fact.** An absolutely convergent series $\sum a_n$ converges.

**Fact.** If a series is absolutely convergent, we can rearrange the terms in the sum in any way we want, and the series will still converge to the same value.
In conditionally convergent series the order of the terms matters! Moreover, for any $q \in \mathbb{R}$ or $q = \pm\infty$ it is possible to rearrange the terms of a conditionally convergent series such that it will converge to $q$.

**Power series**

Let $(c_n)$, $n = 0, 1, 2, \ldots$ be a sequence of numbers. A series of the form

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

is called a *power series* (the $c_n$ are called its *coefficients*).

**Example 6.4.** If $c_n = a$ is a constant, then $P(x) = \sum_{n=0}^{\infty} a x^n = a + ax + ax^2 + \cdots$ is a geometric series. Recall that this converges if $|x| < 1$. For every $x \in \mathbb{R}$ such that $|x| < 1$, we have a series, so we get a function

$$P : (-1, 1) \longrightarrow \mathbb{R}, \qquad P(x) = \sum_{n=0}^{\infty} c_n x^n.$$

The partial sums $s_N = \sum_{n=0}^{N} c_n x^n$ approximate the limit $s = \sum_{n=0}^{\infty} c_n x^n$, so the function $P$ (which is not in general a polynomial) can be approximated by polynomials, as long as $x \in (-1, 1)$, that is, locally in a neighbourhood of 0.

The above example is fundamental in mathematics. The fact that we can approximate many (complicated) functions by polynomials locally around a point is the basis for much of analysis and numerical computations in science.

**Example 6.5.** Recall from (3.4) that for $x$ such that $|x| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1 - x).$$

So, $-\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = P(x)$ and the partial sums are

$$P_N(x) = x + \cdots + \frac{x^N}{N}.$$

Now, $P(x) - P_N(x) = \frac{x^{N+1}}{N+1} + \frac{x^{N+2}}{N+2} + \cdots$, so

$$\frac{P(x) - P_N(x)}{x_{N+1}} = \frac{1}{N+1} + \frac{x}{N+2} + \cdots \longrightarrow \frac{1}{N+1} \quad \text{as } x \longrightarrow 0.$$

Thus, as $x \to 0$, the ratio $\frac{P(x) - P_N(x)}{x_{N+1}}$ is close to $\frac{1}{N+1}$, and therefore

$$P(x) - P_N(x) \text{ is approximated by } \frac{x^{N+1}}{N+1}.$$

We write this

$$P(x) - P_N(x) \sim \frac{x^{N+1}}{N+1}.$$

The difference $P(x) - P_N(x)$ should be thought of as the error when we approximate the function $P(x)$ by the partial sum $P_N(x)$. Clearly, increasing $N$, we can make this error as small as we want.

## Lecture 7

### Convergence of power series

We use the ratio test for $\sum_{n=0}^{\infty} |c_n x^n|$, that is, we look at absolute convergence:

$$p = \lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = \lim_{n\to\infty} |x| \cdot \left| \frac{c_{n+1}}{c_n} \right|.$$

We want to know the values of $x$ for which the series converges.

- If $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = 0$, then $p = 0$ for all $x$, so it converges.
- If $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = c \in \mathbb{R}$, then $p = |x|c$, so $p < 1$ iff $|x| < \frac{1}{c}$. Thus it converges if $|x| < \frac{1}{c}$ and diverges if $|x| > \frac{1}{c}$. If $|x| = \frac{1}{c}$ anything can happen, so we need to check convergence separately.

**Example 7.1.** $P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$. We have $c_n = \frac{1}{n!}$, so

$$\frac{c_{n+1}}{c_n} = \frac{1}{n+1} \longrightarrow 0.$$

Thus the series converges for all $x$. In fact, it converges to $e^x$.

**Example 7.2.** $P(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = -\ln(1-x)$ by (3.4). We have $c_n = \frac{1}{n}$, so

$$\frac{c_{n+1}}{c_n} = \frac{n}{n+1} \longrightarrow 1.$$

Thus the series converges if $|x| < 1$ and diverges if $|x| > 1$. For $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{1}{n+1}$, which is just the harmonic series (diverges!). If $x = -1$, we have the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$, which converges by (6.1) (it's the "alternating harmonic series").

Thus our series converges precisely for $x$ in the half-open interval $[-1, 1)$.

**Definition 7.3.** The set of $x \in \mathbb{R}$ such that $P(x)$ converges is called the *interval of convergence* of the power series. This interval has *radius* $R = \frac{1}{c} = \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|}$ (i.e., half the length of the interval).

*Remark.* We can also use the root test to find the radius of convergence. Namely, $R = \frac{1}{d} = \frac{1}{\lim_{n\to\infty} |c_n|^{1/n}}$ (if the limit exists).

*Remark.* We may also be able to compute the radius (and interval) of convergence if the limits above do not exist.

**Example 7.4.** $P(x) = \sum_{n=0}^{\infty} 2^n x^{2n}$ has all odd coefficients equal to zero, so the ratio test cannot be applied. As $\lim_{n\to\infty} |c_{2n}|^{1/2n} = \sqrt{2} \neq 0$, the root test cannot be applied either.

However, we can write $y = x^2$, and then the series becomes $P(x) = \sum_{n=0}^{\infty} 2^n x^{2n} = \sum_{n=0}^{\infty} 2^n y^n = \sum_{n=0}^{\infty} a_n y^n$, so applying the root test we obtain that the series converges if $|y| < \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|} = \frac{1}{2}$ and diverges if $|y| > 1/2$. Since $y = x^2$, we see that $P(x)$ converges if $|x| < 1/\sqrt{2}$ and diverges if $|x| \geq 1/\sqrt{2}$, so the interval of convergence is $(-1/\sqrt{2}, 1/\sqrt{2})$.

*Remark.* In the case of two-variable analysis (or complex analysis), the convergence will be in a circular disc instead of an interval. This is where the terminology radius of convergence comes from.

For example, $P(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ has interval of convergence $(-\infty, \infty) = \mathbb{R}$ and the radius is $\infty$. The series $P(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$ has interval of convergence $[-1, 1)$ and radius $R = 1$.

**Series with complex numbers**

Assume that we have a power series $P(z) = \sum_{n=0}^{\infty} c_n z^n$, where $z \in \mathbb{C}$ and $c_n \in \mathbb{C}$. We can consider the real series $\sum |c_n z^n|$, which we know converges if

$$|z| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| < 1.$$

Thus, $\sum c_n z^n$ converges aboslutely if $|z| < \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|}$.

**Example 7.5.** $P(z) = \sum_{n=0}^{\infty} z^n$. We have $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = 1$, so the series converges absolutely for $|z| < 1$ and diverges for $|z| > 1$. For $|z| = 1$ it also diverges, because then $z^n$ does NOT tend to 0 (for $z^n$ to tend to 0 its modulus, which is the distance to the origin, would have to tend to 0).

Note that the set of $z \in \mathbb{C}$ such that $|z| < 1$, is a disc (with radius 1), rather than an interval.

**Operations with power series**

Suppose $P(z)$ and $Q(z)$ are two power series which converge in some disc $|z| < R$. Then:

- $P(z) + Q(z)$ and $aP(z)$ converge, for any $a \in \mathbb{C}$.
- Inside the disc of convergence, we can integrate and differentiate series.

Why can we differentiate series inside the interval/disc of convergence? Well, if $P(z) = \sum c_n z^n$, we know that the radius is

$$R = \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|}.$$

Now, $\frac{dP(z)}{dz} = \sum n c_n z^{n-1}$ and the radius of this series is

$$\frac{1}{\lim_{n\to\infty} \left| \frac{(n+1)c_{n+1}}{n c_n} \right|} = \frac{1}{\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|} = R,$$

so the radius of convergence is the *same* as before. This means that taking the derivative does not "destroy" the disc of convergence.

**Example 7.6.** $P(x) = \sum x^n = \frac{1}{1-x}$ for $|x| < 1$. Thus,

$$\begin{aligned} \frac{dP(x)}{dx} &= (x-1)^{-2} = (1 + x + x^2 + \cdots)' = 1 + 2x + 3x^2 + \cdots \\ &= \sum_{n=1}^{\infty} n x^{n-1} = \sum_{k=0}^{\infty} (k+1) x^k. \end{aligned}$$

This series is called the *expansion* of $(x-1)^{-2} = \frac{1}{(x-1)^2}$ (at $x = 0$). In the following, we will now see what this means in general.

## Lecture 8

### Taylor series

Many (most?) physical phenomena cannot be completely described by polynomial functions (because these are too simple to do the job, in general). However, if we study the behaviour of functions in a small region around a point at a time, then we can approximate the function well by polynomials. This has big advantages from a computational point of view, because to compute polynomials we only need the four standard operations (no square roots, logarithms etc are needed).

**Goal:** To approximate a function by polynomials which are partial sums of a power series (in some interval around a point).

Recall that if we have a function $y = f(x)$, we can plot its graph in the $x-y$-plane. We want to approximate the graph by something simple. The simplest graph is a line. We can then approximate the graph of $f$ *at a point* $a$ by taking the tangent at $a$. The slope of this tangent line is of course the derivative $f'(x)$, and the equation for the line is

$$y = f'(a)(x-a) + f(a).$$

This is a first approximation of $f$ and it is called the *first Taylor polynomial* $p_1(x)$.

Why is it a "good" approximation? Well, the error is

$$f(x) - p_1(x) = f(x) - f(a) - f'(a)(x-a) = (x-a)\underbrace{\left(\frac{f(x)-f(a)}{x-a} - f'(a)\right)}_{\to 0 \text{ as } x \to a}.$$

So, as we approach the point $x = a$, the error goes to 0.

We now want to find better approximations, so-called "higher" Taylor polynomials $p_2(x), p_3(x), \ldots$ such that for each $n$

$$\frac{f(x) - p_n(x)}{(x-a)^n} \longrightarrow 0 \text{ as } x \longrightarrow a.$$

Let $x$ be approximately equal to $a$. We write this $x \approx a$. We can write

$$\int_a^x f'(t)dt = f(x) - f(a),$$

and so

$$f(x) = f(a) + \int_a^x f'(t)dt. \tag{☺}$$

Apply the equality above to the function $f'$ at point $t \in (a, x)$ (or $(x, a)$):

$$f'(t) = f'(a) + \int_a^t f''(s)ds \approx f'(a) + (t-a)f''(a).$$

Plug this into ☺ :

$$\begin{aligned} f(x) &\approx f(a) + \int_a^x \left(f'(a) + (t-a)f''(a)\right) dt \\ &= f(a) + f'(a)(x-a) + f''(a)\int_a^x (t-a)dt \\ &= f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} = p_2(x). \end{aligned}$$

This is our second Taylor polynomial $p_2(x)$.

**Definition 8.1.** In general, we get the $N$th *Taylor polynomial*:

$$\begin{aligned} p_N(x) &= f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \frac{(x-a)^3}{3!}f^{(3)}(a) \\ &\quad + \cdots + \frac{(x-a)^N}{N!}f^{(N)}(a) \\ &= \sum_{n=0}^{N} \frac{(x-a)^n}{n!} f^{(n)}(a). \end{aligned}$$

(note that $0! = 1$).

**Example 8.2.** $f(x) = x^2 + 2x + 1$ and $a = 0$. Then

$$\begin{aligned} f(a) &= 1 & & \\ f'(a) &= 2 & p_1(x) &= 1 + 2x \\ f''(a) &= 2 & p_2(x) &= 1 + 2x + \frac{2x^2}{2} = f(x) \\ f^{(3)}(a) &= 0 & p_3(x) &= p_2(x) \\ \cdots &= 0 & &\cdots \\ \cdots &= 0 & p_N(x) &= p_2(x). \end{aligned}$$

In general, the $N$th Taylor polynomials of a polynomial function $f(x)$ of degree $n$ will eventually (for $N \geq n$) be the polynomial $f(x)$ itself.

Here are some other examples of Taylor polynomials:

**Example 8.3.** Let $f(x) = \sin x$. Compute $p_5(x)$, around $a = 0$.

| | Values at $a = 0$ |
|---|---|
| $f(x) = \sin x$ | 0 |
| $f'(x) = \cos x$ | 1 |
| $f''(x) = -\sin x$ | 0 |
| $f^{(3)}(x) = -\cos x$ | $-1$ |
| $f^{(4)}(x) = \sin x$ | 0 |
| $f^{(5)}(x) = \cos x$ | 1. |

Thus $p_5(x) = 0 + x + \frac{x^2}{2} \cdot 0 + \frac{x^3}{3!}(-1) + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

Remark: Instead of a Taylor series "around" a point $a$, some authors say "centered at $a$", "about $a$", or simply "at $a$". The point is (no pun intended) that the Taylor expansion is only valid in sufficiently small neighbourhoods around the point $a$.

**Example 8.4.** Let $f(x) = \ln x$. Compute $p_4(x)$ around $a = 1$.

| | Values at $a = 1$ |
|---|---|
| $f(x) = \ln x$ | 0 |
| $f'(x) = \frac{1}{x}$ | 1 |
| $f''(x) = -\frac{1}{x^2}$ | $-1$ |
| $f^{(3)}(x) = \frac{2}{x^3}$ | 2 |
| $f^{(4)}(x) = \frac{-3!}{x^4}$ | $-3!$ |

Thus $p_4(x) = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$.

Setting $y = x - 1$, we get the Taylor polynomial $p_4(y)$ for $f(y) = \ln(1 + y)$ around $a = 0$:

$$y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4}.$$

## Lecture 9

**Definition 9.1.** The *Taylor series* of $f(x)$ around $a$ is a power series

$$\sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}.$$

Each partial sum is a Taylor polynomial. (For $a = 0$ this is also called a Maclaurin series.)

Note that the Taylor series is defined only if $f(x)$ is infinitely differentiable at $x = a$ (i.e., if $f^{(n)}(a)$ exists for all $n \geq 1$). If the Taylor series converges, it will equal the value of $f(x)$ for $x$ close to $a$. Further away from $a$ it may not approximate $f(x)$ well. For some nice functions, the Taylor series equals $f(x)$ everywhere:

**Example 9.2.** The Taylor series of $e^x$ and $\sin x$ around $a = 0$ are

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

Note that these converge absolutely for all $x \in \mathbb{R}$.

By contrast, the Taylor series for $\ln(1 + x)$ does not converge everywhere:

**Example 9.3.** In (8.4) we see a recurring pattern in the partial sums, so

$$\ln(1+y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} y^n$$

(we could use $x$ instead of $y$ here). For convergence,

$$\left| \frac{\frac{(-1)^{n+2}}{(n+1)} y^{n+1}}{\frac{(-1)^{n+1}}{n} y^n} \right| = |y| \cdot \frac{n}{(n+1)} \longrightarrow |y|,$$

so it converges if $|y| < 1$. For $|y| > 1$ it diverges and is therefore *not* a good approximation.

*Conclusion.* A Taylor series (or any power series) converges absolutely in a disc/interval about its expansion point $a$, and diverges outside of it.

**Theorem 9.4** (Taylor's theorem)**.** *Suppose $f$ is $N + 1$ times differentiable in a neighborhood of $a$, $x$ belongs to this neighborhood, and $f^{(N+1)}$ is continuous between $a$ and $x$. Then*

$$f(x) = p_N(x) + \underbrace{\frac{f^{(N+1)}(t)}{(N+1)!}(x-a)^{N+1}}_{=R_N(x)},$$

*for some $t$ such that $a < t < x$ (or $x < t < a$). Here $p_N(x)$ is the $N$th Taylor polynomial and $R_N(x)$ is the remainder/error in* Lagrange form*.*

By Taylor's theorem, we can bound the error when $f(x)$ is approximated by $p_N(x)$:

**Example 9.5.** Compute $\cos(0.2)$ with error less than $10^{-4}$. Plan: Let $f(x) = \cos x$ and $p_N(x)$ be the Taylor polynomial around $a = 0$. Find $N$ such that

$$|\cos(x) - p_N(x)| < 10^{-4},$$

and then compute $p_N(0.2)$.

The error term in Lagrange form is

$$R_N(x) = \frac{f^{(N+1)}(t)}{(N+1)!} x^{N+1}.$$

The derivatives $f^{(n)}(x)$ of $f(x) = \cos x$ are all either $\pm \sin x$ or $\pm \cos x$. Thus, $|f^{(N+1)}(t)| \leq 1$, and so

$$|R_N(x)| \leq \frac{|x|^{N+1}}{(N+1)!}.$$

This implies (by Taylor's theorem) that

$$|\cos x - p_N(x)| \leq \frac{|x|^{N+1}}{(N+1)!}$$

and hence, setting $x = 0.2$,

$$|\cos(0.2) - p_N(0.2)| \leq \frac{|0.2|^{N+1}}{(N+1)!}.$$

Now find $N$ such that $\frac{|0.2|^{N+1}}{(N+1)!} \leq 10^{-4}$. Try $N = 3$:

$$\frac{(2 \cdot 10^{-1})^{3+1}}{(3+1)!} = 16 \cdot 10^{-4}/24 < 10^{-4},$$

so this will work! We just need to evaluate $p_3(0.2)$:

$$p_3(x) = 1 + x(-\sin 0) + \frac{x^2}{2}(-\cos 0) + \frac{x^3}{3!}(\sin 0) = 1 - \frac{x^2}{2},$$

so

$$p_3(0.2) = 1 - 0.02 = 0.98.$$

Thus,

$$\cos(0.2) \approx 0.98,$$

and this value has error at most $10^{-4}$, that is, it is correct to at least four decimal places.

Check with a calculator (using radians, not degrees, for angles): $\cos(0.2) = 0.980066\ldots$

# Lecture 10

## Taylor series and limits

Taylor series can be used to calculate limits of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)},$$

where both $f(x)$ and $g(x)$ go to 0 as $x \to a$. Sometimes, but not always, l'Hôpital's rule can also be used.

**Example 10.1.** Compute $\lim_{x\to 0} \frac{\sin x}{x}$. First consider $\sin x$ for $x$ near 0. By (8.3), we can write

$$\sin x = x - \frac{x^3}{3!} + o(x^3),$$

where $o(x^3)$ denotes terms which are of order $x^4$ and higher (because higher powers like $x^4$ and further go to 0 faster than $x^3$, i.e. $\lim_{x\to 0} \frac{o(x^3)}{x^3} = 0$). Then we can write

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3} + o(x^3)}{x} = \lim_{x\to 0} \frac{x(1 - \frac{x^2}{3} + \frac{o(x^3)}{x})}{x} = \lim_{x\to 0} 1 - \frac{x^2}{3} + \frac{o(x^3)}{x} = 1,$$

since

$$\lim_{x\to 0} \frac{o(x^3)}{x} = \lim_{x\to 0} \frac{o(x^3)}{x^3} x^2 = \lim_{x\to 0} \frac{o(x^3)}{x^3} \lim_{x\to 0} x^2 = 0 \cdot 0 = 0.$$

*Remark.* Note that the $o(x^n)$ does not denote a concrete function, but any (converging) power series $\sum a_k x^k$ around 0 such that the coefficients $a_0, \dots a_n$ are all equal to zero (as this guarantees that $\lim_{x\to 0} \frac{\sum a_k x^k}{x^n} = 0$). For example, $o(x^5)$ is simultaneously $o(x^3)$ as $\lim_{x\to 0} \frac{o(x^5)}{x^5} = 0$ implies

$$\lim_{x\to 0} \frac{o(x^5)}{x^3} = \lim_{x\to 0} \frac{o(x^5)}{x^5} x^2 = \lim_{x\to 0} \frac{o(x^5)}{x^5} \lim_{x\to 0} x^2 = 0 \cdot 0 = 0.$$

(but the converse may not be true of course!), $\frac{o(x^n)}{x} = o(x^{n-1})$ (as we can see in the example above for $n = 3$), $o(x^n) \cdot x^m = o(x^{n+m})$ for general $m, n$, and $o(x^n) + o(x^n) = o(x^n)$.

Here is an example where we cannot use l'Hôpital's rule, and where Taylor series works (see Q22 from the problem sheet)

**Example 10.2.** Let

$$f(x) = \exp\left(\frac{\sin x}{1 - 3x}\right)$$

and calculate

$$\lim_{x\to 0} \frac{f(x) - (x+1)}{x \cos x - \ln(1+x)}. \tag{limit}$$

Trying l'Hôpital leads to a horrible mess which is not easier than the original function. Use Taylor series instead.

First consider $f(x)$ for $x$ near 0 ($|x| < 1$). By (8.3) and the geometric series $\frac{1}{1-x} = 1+x+x^2+\cdots$ for $|x| < 1$, we can write

$$\sin x = x - \frac{x^3}{3!} + o(x^3), \qquad \text{and} \qquad \frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + o(x^3),$$

Thus

$$\begin{aligned}\frac{\sin x}{1-3x} &= \left(x - \frac{x^3}{3!} + o(x^3)\right)(1 + 3x + 9x^2 + 27x^3 + o(x^3)) \\ &= x - \frac{x^3}{3!} + o(x^3) + 3x^2 + 9x^3 + o(x^3) \\ &= x + 3x^2 + \frac{53}{6}x^3 + o(x^3).\end{aligned}$$

We also know that $e^x = 1 + x + \frac{x^2}{2} + \cdots$ and plugging in the above, we get

$$\begin{aligned}f(x) = 1 + x + 3x^2 + \frac{53}{6}x^3 + o(x^3) + \left(x + 3x^2 + \frac{53}{6}x^3 + o(x^3)\right)^2 /2 \\ + \left(x + 3x^2 + \frac{53}{6}x^3 + o(x^3)\right)^3 /6 + o(x^3) \\ = 1 + x + 3x^2 + \frac{53}{6}x^3 + o(x^3) + x^2/2 + (2x \cdot 3x^2)/2 + x^3/6 + o(x^3) \\ = 1 + x + \frac{7}{2}x^2 + 12x^3 + o(x^3).\end{aligned}$$

To calculate the final limit, we also need to Taylor expand $\cos x\ \ln(1+x)$ up to order 3:

$$\cos x = 1 - \frac{x^2}{2} + o(x^3), \qquad \text{and} \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3),$$

so $x\cos x - \ln(1+x) = \frac{x^2}{2} - \frac{5}{6}x^3 + o(x^3)$. Thus,

$$(\text{limit}) = \lim_{x\to 0} \frac{\frac{7}{2}x^2 + 12x^3 + o(x^3)}{\frac{x^2}{2} - \frac{5}{6}x^3 + o(x^3)} = \frac{\frac{7}{2} + 12x + o(x)}{\frac{1}{2} - \frac{5}{6}x + o(x)} = \frac{7/2}{1/2} = 7.$$

Another example of computing limits.

**Example 10.3.**

$$\lim_{x\to 0} \frac{\frac{x^2}{1-x} + 2\cos x - 2}{2x^3 + 3x^7} = \lim_{x\to 0} \frac{x^2(\sum x^k) + 2\cos x - 2}{2x^3 + 3x^7} =$$

$$\lim_{x\to 0} \frac{x^2 + x^3 + o(x^3) + 2(1 - \frac{x^2}{2} + o(x^3)) - 2}{2x^3 + 3x^7} = \lim_{x\to 0} \frac{x^2 + x^3 + o(x^3) - x^2 + o(x^3)}{2x^3 + o(x^3)} =$$

$$\lim_{x\to 0} \frac{x^3 + o(x^3)}{2x^3 + o(x^3)} = \lim_{x\to 0} \frac{1 + \frac{o(x^3)}{x^3}}{2 + \frac{o(x^3)}{x^3}} = \frac{\lim_{x\to 0} 1 + \frac{o(x^3)}{x^3}}{\lim_{x\to 0} 2 + \frac{o(x^3)}{x^3}} = \frac{1 + \lim_{x\to 0} \frac{o(x^3)}{x^3}}{2 + \lim_{x\to 0} \frac{o(x^3)}{x^3}} = \frac{1+0}{2+0} = 1/2.$$

# Lecture 11

## Matrices

Although the mathematical functions describing natural phenomena can be very complicated, one can say that locally, everything behaves linearly. Think of the tangent of a curve at a point: it is a *line* which approximates the function locally around a point. This is one reason why *linear algebra* is so useful.

Some of the main players in linear algebra are matrices. Matrices help to streamline the solution to system of linear equations:

**Example 11.1.** Solve the system

$$\begin{cases} x + 2y &= 1 \qquad (1) \\ -2x - 3y &= 2 \qquad (2). \end{cases}$$

To solve this we transform the equations: First, equation (2) is transformed into "equation (1) added twice to equation (2)", that is

$$(2) \longrightarrow 2 \cdot (1) + (2).$$

This gives the new system

$$\begin{cases} x + 2y &= 1 \qquad (1) \\ y &= 4 \qquad (2). \end{cases}$$

The point here was to cancel all $x$s in (2). Now cancel $y$ in (1) via

$$(1) \longrightarrow (1) - 2 \cdot (2)$$

to obtain

$$\begin{cases} x &= -7 \qquad (1) \\ y &= 4 \qquad (2). \end{cases}$$

We have now obtained the solution.

As the above example demonstrates, what matters is not the variables $x$ and $y$ themselves, but only the coefficients of the equations, that is the tables of numbers

$$\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

which encode the system of equations. These tables of numbers are called *matrices*. In general, a matrix is a rectangular array of numbers, called its *entries*. A matrix is said to be an $m \times n$ matrix if it has $m$ rows and $n$ columns. For example,

$$\begin{pmatrix} 3 & \frac{1}{2} \\ -1 & 0 \\ 57 & \pi \end{pmatrix}$$

is a $3 \times 2$ matrix.

Here are some important special cases of matrices:

- $n = 1$: A $m \times 1$ matrix $\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$ is a *column vector* (or just vector).
- $m = 1$: A $1 \times m$ matrix $(\cdots\cdots)$ is a *row vector*.
- $n = m$: An $n \times n$ matrix is a *square matrix*.

We can write a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

where the $a_{ij}$ are the entries, for $i = 1, 2, 3$ and $j = 1, 2$. This is a $3 \times 2$ matrix, but we can extend this notation to any $m \times n$ matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

We can write this compactly as $(a_{ij})$, remembering that $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

The entries $a_{ij}$ can be real of complex numbers. Assume first that $a_{ij} \in \mathbb{R}$. We then let

$$\mathrm{Mat}_{m\times n}(\mathbb{R})$$

be the set of all $m \times n$ matrices with real entries, and

$$\mathrm{Mat}_n(\mathbb{R})$$

be the set of $n \times n$ square matrices.

**Matrix multiplication**

There is a way to multiply two matrices, which is a bit unusual at first, but turns out to be very useful.

**Example 11.2.** Let $A$ and $B$ be the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

Then the product is

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 5 + 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}.$$

Note that the product of the square matrix $A$ with the vector matrix $B$ is another vector matrix.

In general, a product of $m \times n$ matrix $A$ and an $n \times l$ matrix $B$ is $m \times l$ matrix $C$, and

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

The reason why it is useful to define matrix multiplication like this is that we can write the system of equations in (11.1) as

$$\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ -2x-3y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We have therefore replaced a system with *two* equations by a *single* matrix equation. This is helpful if we had a system of 1000 equations, especially if we want to solve it using a computer (which, surely, we want).

We can also multiply two square matrices:

**Example 11.3.** Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1\cdot 0 + 2\cdot 0 & 1\cdot 2 + 2\cdot 0 \\ 3\cdot 0 + 4\cdot 0 & 3\cdot 2 + 4\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 6 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0\cdot 1 + 2\cdot 3 & 0\cdot 2 + 2\cdot 4 \\ 0\cdot 1 + 0\cdot 3 & 0\cdot 2 + 0\cdot 4 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 0 & 0 \end{pmatrix}.$$

So we see that $AB$ is *not* equal to $BA$! Moreover,

$$BB = B^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which is called the zero matrix $0_{2\times 2}$ (or simply 0). We see that it can happen that the square of a non-zero matrix is zero!

We can multiply a $2\times 2$ matrix with a $2\times 1$ one, but not with a $3\times 1$ or bigger vector matrix. In general, we can multiply an $m\times n$ matrix by an $n\times k$ one. For example, a $2\times 3$ one by a $3\times 2$ one:

$$\begin{pmatrix} 1 & -4 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + (-4)(-1) + 3\cdot 0 & 1\cdot 2 + (-4)\cdot 0 + 3\cdot 5 \\ 0\cdot 3 + 0\cdot(-1) + 1\cdot 0 & 0\cdot 2 + 0\cdot 0 + 1\cdot 5 \end{pmatrix} = \begin{pmatrix} 7 & 17 \\ 0 & 5 \end{pmatrix}.$$

Similarly, we have

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = (1\cdot 3 + 2\cdot 4) = (11),$$

and

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 & 3\cdot 2 \\ 4\cdot 1 & 4\cdot 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}.$$

**Summary**

- If $A \in \text{Mat}_{m\times n}(\mathbb{R})$ and $B \in \text{Mat}_{n\times k}(\mathbb{R})$, we can multiply $A$ and $B$ and $AB \in \text{Mat}_{m\times k}(\mathbb{R})$.
- We can only multiply two matrices $A$ and $B$ if $A$ has the same number of columns as $B$ has rows.
- We can have $AB \neq BA$. If $AB = AB$ (which happens sometimes) the matrices $A$ and $B$ are said to *commute*.
- We can have $AB = 0$, even though $A \neq 0$ and $B \neq 0$.

## Other operations on matrices

- Addition of matrices is easy. Just add element-wise: $(A+B)_{ij} = a_{ij} + b_{ij}$. For example,

  $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+0 & 2+2 \\ 3+0 & 4+0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix}.$$

  Since $a + b = b + a$ for any real numbers $a, b$, it is clear that $A + B = B + A$, for two matrices $A, B$.
  Note that we can only add two matrices if they are of the *same* size.

- If $\lambda \in \mathbb{R}$ is a scalar and $A = (a_{ij}) \in \text{Mat}_{m\times n}(\mathbb{R})$, then $\lambda A = (\lambda a_{ij})$. For example, if $\lambda = 2$ and $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, we have

  $$2 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}.$$

- If $A$ is a matrix, we can turn its rows into columns (and columns into rows; same thing). The result is called the *transpose*: $A^T$ of $A$, for example:

  $$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

- If $A$ is an $m \times n$ matrix, then $-A$ is the matrix $0_{m\times n} - A$, where $0_{m\times n}$ is the zero matrix of that size. In other words, to get $-A$ just change sign on each of the entries of $A$.

# Lecture 12

**Definition 12.1.** A matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ is called *symmetric* if $A = A^T$, for example $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$. A square matrix is called *diagonal* if all its off-diagonal entries are zero, for example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Note that a diagonal matrix is always symmetric.

### An application of matrices

**Example 12.2.** Suppose I want to compute the nutritional value of what someone eats throughout a week. We can encode the data in matrices, for easy computation:

$$\text{Nutrients: } D = \begin{array}{c} \begin{matrix} \text{pizza} & \text{beer} \end{matrix} \\ \begin{pmatrix} 282 & 140 \\ 13 & 0 \\ 7 & 1 \end{pmatrix} \end{array} \begin{matrix} \text{kcal} \\ \text{fat} \\ \text{protein} \end{matrix},$$

$$\text{Quantity: } W = \begin{array}{c} \begin{matrix} \text{M} & \text{T} & \text{W} & \text{Th} & \text{F} & \text{Sa} & \text{Su} \end{matrix} \\ \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 5 & 0 & 0 \end{pmatrix} \end{array} \begin{matrix} \text{pizza} \\ \text{beer} \end{matrix}.$$

If we take the matrix product $P = DW$, we get a $3 \times 7$ matrix

$$P = (p_{ij}) = \begin{array}{c} \begin{matrix} \text{M} & \text{T} & \text{W} & \text{Th} & \text{F} & \text{Sa} & \text{Su} \end{matrix} \\ \begin{pmatrix} 704 & 0 & 0 & 0 & 982 & 0 & 0 \\ 26 & 0 & 0 & 0 & 13 & 0 & 0 \\ 15 & 0 & 0 & 0 & 12 & 0 & 0 \end{pmatrix} \end{array} \begin{matrix} \text{kcal} \\ \text{fat} \\ \text{protein} \end{matrix}.$$

So, for example, the total calorie intake on Monday is $p_{11} = 704$ and the total fat intake on Friday is $p_{25} = 13$.

The point is that the matrix $D$ always stays as a constant, even if the weekly eating habits change, so this is a convenient way of encoding and computing such data. The matrix $P$ encodes everything we want in a neat form.

### Matrices and series

Let $A$ be an $m \times m$ matrix (i.e., a square one). For any integer $n = 1, 2, 3, \ldots$ we can compute the $n$th power of $A$:

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}.$$

Moreover, we define $A^0 = I_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, which is called the *identity matrix* of size $m$.

So, if we have a polynomial $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$, then we can evaluate it on the matrix $A$:

$$p(A) = c_0 I_m + c_1 A + c_2 A^2 + \cdots + c_n A^n.$$

The value $p(A)$ is still an $m \times m$ matrix.

Now, let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series. We define

$$f(A) = \sum_{n=0}^{\infty} c_n A^n.$$

This series may or may not converge to a matrix.

**Example 12.3.** Let $A = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$. Then

$$A^2 = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$A^3 = \begin{pmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

So any higher power of $A$ is also 0. Thus, we can compute any power series of $A$, for example, the exponent:

$$\begin{aligned} \exp(A) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n &= I_3 + A + \frac{1}{2} A^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & xy/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x & xy/2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

**Example 12.4.** If we have a diagonal matrix, things are easy. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then

$$A^2 = \begin{pmatrix} 1^2 & 0 \\ 0 & 2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 2^3 \end{pmatrix}, \ldots$$

so we have $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = \begin{pmatrix} e^1 & 0 \\ 0 & e^2 \end{pmatrix}$.

## Systems of linear equations

We now return to one of the original motivations for matrices. We will explain how matrices can be used to efficiently solve systems of linear equations.

**Example 12.5.** We solve system by successive transformations:

$$\begin{cases} y+3z &= -1 \\ x+y+2z &= 1 \\ 2x+y &= 2 \end{cases} \xrightarrow{R_1 \leftrightarrow R_2} \begin{cases} x+y+2z &= 1 \\ y+3z &= -1 \\ 2x+y &= 2 \end{cases} \quad \begin{array}{c} (R_i \text{ denotes row } i) \\ \text{(the first step just swaps rows 1 and 2)} \end{array}$$

$$\xrightarrow{R_3-2R_1} \begin{cases} x+y+2z &= 1 \\ y+3z &= -1 \\ -y-4z &= 0 \end{cases} \text{(eliminates } x \text{ from row 3)}$$

$$\xrightarrow{R_3+R_2} \begin{cases} x+y+2z &= 1 \\ y+3z &= -1 \\ -z &= -1 \end{cases} \text{(eliminates } y \text{ from row 3)} \xrightarrow{-R_3} \begin{cases} x+y+2z &= 1 \\ y+3z &= -1 \\ z &= 1 \end{cases}$$

$$\xrightarrow{R_2-3R_3} \begin{cases} x+y+2z &= 1 \\ y &= -4 \\ z &= 1 \end{cases} \text{(eliminates } z \text{ from row 2)}$$

$$\xrightarrow{R_1-2R_3} \begin{cases} x+y &= -1 \\ y &= -4 \\ z &= 1 \end{cases} \text{(eliminates } z \text{ from row 1)}$$

$$\xrightarrow{R_1-R_2} \begin{cases} x &= 3 \\ y &= -4 \\ z &= 1 \end{cases} \text{(eliminates } y \text{ from row 1).}$$

To describe a streamlined algorithm for solving systems, we reformulate the problem in terms of matrices. The system above can be written

$$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

If we let $A$ be the $3 \times 3$ matrix, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, then we can just write this as $A\underline{x} = \underline{b}$.

What did we do when we solved the system in (12.5)? We only transformed $A$ and $\underline{b}$ in seven steps:

**Step 1** Swap two eqns/rows.

**Steps 2,3,5,6,7** Add a multiple of an eqn to another eqn.

**Step 4** Multiply an eqn by a number.

When we do something to the equations, we can do the same to the rows of the matrices $A$ and $\underline{b}$. So, we create the *augmented matrix* of the system

$$(A|\underline{b}) = \left( \begin{array}{ccc|c} 0 & 1 & 3 & -1 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 2 \end{array} \right).$$

We can now perform the seven steps above in terms of this matrix alone. For example, Steps 1-2 would be

$$\left(\begin{array}{ccc|c} 0 & 1 & 3 & -1 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 2 & 1 & 0 & 2 \end{array}\right) \xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & -1 & -4 & 0 \end{array}\right).$$

**Elementary row operations**

The operations we have used in Steps 1-7 above are called *elementary row operations (ERO).* The imporant thing to note is that they do *not* change the solutions to the system. The goal is to obtain the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

since this gives us $x = 3$, $y = -4$, $z = 1$.

We will now use matrices to solve another system:

**Example 12.6.** Solve the system

$$\begin{cases} x + y & = & 1 \\ x - y & = & 2 \\ 2x - y & = & 3 \end{cases}.$$

We write the augmented matrix and perform ERO on it:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 2 & -1 & 3 \end{array}\right) \xrightarrow{R_3 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -3 & 1 \end{array}\right)$$

$$\xrightarrow[-\frac{1}{3}R_3]{-\frac{1}{2}R_2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 1 & -1/3 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{array}\right).$$

The last row means that $0 \cdot x + 0 \cdot y = 1/6$, which is impossible. Thus the system has *no* solutions at all.

# Lecture 13

**Example 13.1.** Here is another situation with three variables and three equations.

$$\left\{\begin{array}{rcl} x+y &=& 3\\ y+z &=& 1\\ x-z &=& 2\end{array}\right. \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3\\ 0 & 1 & 1 & 1\\ 1 & 0 & -1 & 2\end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3\\ 0 & 1 & 1 & 1\\ 0 & -1 & -1 & -1\end{array}\right)$$

$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{array}\right)$$

$$\xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{array}\right).$$

This is equivalent to the system

$$\left\{\begin{array}{rcl} x-z &=& 2\\ y+z &=& 1\end{array}\right. \iff \left\{\begin{array}{rcl} x &=& 2+z\\ y &=& 1-z\end{array}\right. .$$

For any $z \in \mathbb{R}$ (without restriction) we thus have a solution

$$(x, y, z) = (2+z, 1-z, z).$$

Hence, there are *infinitely many* solutions to this system.

**Example 13.2.**

$$\left\{\begin{array}{rcl} 2x+2y+3z &=& 7\\ x+2y-z &=& 0\end{array}\right. \longrightarrow \left(\begin{array}{ccc|c} 2 & 2 & 3 & 7\\ 1 & 2 & -1 & 0\end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|c} 0 & -2 & 5 & 7\\ 1 & 2 & -1 & 0\end{array}\right)$$

$$\xrightarrow{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0\\ 0 & -2 & 5 & 7\end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 7\\ 0 & -2 & 5 & 7\end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 7\\ 0 & 1 & -5/2 & -7/2\end{array}\right) \longrightarrow \left\{\begin{array}{rcl} x+4z &=& 7\\ y-5z/2 &=& -7/2\end{array}\right. .$$

$$\iff \left\{\begin{array}{rcl} x &=& 7-4z\\ y &=& -7/2+5z/2\end{array}\right. .$$

So for each $z \in \mathbb{R}$ we have a solution for $x$ and $y$. Thus the system has infinitely many solutions.

## Gauss elimination

This is the general algorithm to solve a linear system of equations. We have already seen the method in several examples, but we will describe exactly what its goal is, and give more examples.

In order to solve a system $(A \mid \underline{b})$, we use ERO to transform this matrix into one where $A$ is *as close as possible* to an identity matrix. Looking back at some of the previous examples, we got:

- Example (12.5):
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right).$$
(unique solution)

- Example (13.1):
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$
(infinitely many solutions)

It is not always possible to achieve an identity matrix, but something close to it is always possible, namely a matrix such that:

- Every row starts with some 0-entries (possibly no 0-entries for the first row) followed by a 1-entry (unless the whole row is zero).
- The number of 0-entries in a row is *more* than in the preceding row (unless both the row and the preceding one consist of 0-entries only).
- The first non-zero entry in each row lies below 0-entries of all previous rows.

**Definition 13.3.** Augmented matrices satisfying the above three conditions are said to be in *Reduced Row Echelon Form (RREF).*

In (13.4), the matrix
$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
is in RREF. We have no zeros in the beginning of the 1st row, two zeros in the 2nd, and three in the 3rd. Note that the first non-zero entry in the 2nd row lies below a 0-entry of the 1st row.

Here is an example to illustrate RREF:

**Example 13.4.**

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 3 & -1 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 2 & 3 & -1 \end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right)$$
$$\xrightarrow{-R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{array}\right) \xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\iff \left\{ \begin{array}{rcl} x & = & 2-2y \\ z & = & -1. \end{array}\right.$$

Thus, we have have infinitely many solutions (one for each $y \in \mathbb{R}$).

Here is another example, which shows how to apply the ERO systematically in Gaussian elimination to achieve RREF.

**Example 13.5.**

$$\left\{\begin{array}{rcr} x+2y+z &=& 3\\ 2x+y-z &=& -3\\ x+y+2z &=& 4\\ 2x+3y+3z &=& 7\end{array}\right. \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 2 & 1 & -1 & -3\\ 1 & 1 & 2 & 4\\ 2 & 3 & 3 & 7\end{array}\right) \xrightarrow[R_4-2R_1]{\substack{R_2-2R_1\\ R_3-R_1}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & -3 & -3 & -9\\ 0 & -1 & 1 & 1\\ 0 & -1 & 1 & 1\end{array}\right)$$

(we did this to clear the entries below the first 1-entry)

$$\xrightarrow{-\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & 1 & 1 & 3\\ 0 & -1 & 1 & 1\\ 0 & -1 & 1 & 1\end{array}\right) \xrightarrow[R_4+R_2]{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & 1 & 1 & 3\\ 0 & 0 & 2 & 4\\ 0 & 0 & 2 & 4\end{array}\right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & 1 & 1 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 2 & 4\end{array}\right)$$

$$\xrightarrow{R_4-2R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & 1 & 1 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{array}\right) \xrightarrow[R_2-R_3]{R_1-R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{array}\right).$$

This last matrix is in RREF, and the matrix to the left of the vertical line is an identity matrix with an extra zero-row added. This means that the system has a *unique* solution:

$$x = -1, \quad y = 1, \quad z = 2.$$

# Lecture 14

## Structure of solutions of linear systems

The discussion and examples so far show that three cases can appear:

- A unique solution: The RREF looks like an identity matrix with extra zero-rows:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{array}\right).$$

- No solutions: The RREF has a row with only zeros on the left and a non-zero entry on the right.

- Infinitely many solutions: The number of rows in the RREF is *smaller* than the number of columns (when zero rows are ignored).

A linear system

$$A\underline{x} = \underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

is called *homogeneous*. The set of solutions is then called the *kernel* of the matrix $A$. There is always at least one solutions: $\underline{x} = \underline{0}$. If the RREF of $(A \mid \underline{0})$ has less rows than columns, then the system $A\underline{x} = \underline{0}$ also has non-zero solutions.

Now, suppose we have a linear system

$$A\underline{x} = \underline{b},$$

and let $\underline{x}_1$ and $\underline{x}_2$ be two solutions, that is, $A\underline{x}_1 = \underline{b}$ and $A\underline{x}_2 = \underline{b}$. Then

$$A(\underline{x}_1 - \underline{x}_2) = \underline{b} - \underline{b} = \underline{0}.$$

Thus the difference of any two solutions is a solutions to a homogeneous system. In particular, if we fix one solution $\underline{x}_0$ such that $A\underline{x}_0 = \underline{b}$, then *any* solution of $A\underline{x} = \underline{b}$ has the form

$$\underline{x} = \underline{x}_0 + \underline{x}_h,$$

where $\underline{x}_h$ runs through all the solutions to the homogeneous system $A\underline{x} = \underline{0}$.

Thus, to solve the system $A\underline{x} = \underline{b}$, we need to know:

- *one* solution to $A\underline{x} = \underline{b}$,

- all the solutions to $A\underline{x} = \underline{0}$.

**Example 14.1.**

$$\begin{cases} x+y &= 1 \\ 2x+2y &= 2 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \Longleftrightarrow x+y=1.$$

So one solution is, for example, $\underline{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now, the solutions of $A\underline{x} = \underline{0}$ are given by

$$x+y=0,$$

that is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Thus, the solutions of the original system are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ 1-\lambda \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R}.$$

**Systems with a parameter**

**Example 14.2.** Find the values of $\lambda \in \mathbb{R}$ such that the following system has no solutions, one solution, or infinitely many solutions:

$$\begin{cases} x+\lambda y+z &= 1 \\ \lambda x+y+(\lambda-1)z &= \lambda \\ x+y+z &= \lambda+1 \end{cases}.$$

We use ERO to put matrix of the system into RREF:

$$\left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ \lambda & 1 & \lambda-1 & \lambda \\ 1 & 1 & 1 & \lambda+1 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-\lambda R_1} \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1-\lambda^2 & -1 & 0 \\ 0 & 1-\lambda & 0 & \lambda \end{array}\right) \xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1-\lambda & 0 & \lambda \\ 0 & 1-\lambda^2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3-(1+\lambda)R_2} \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1-\lambda & 0 & \lambda \\ 0 & 0 & -1 & -\lambda(1+\lambda) \end{array}\right) \xrightarrow[-R_3]{R_1+R_3} \left(\begin{array}{ccc|c} 1 & \lambda & 0 & 1-\lambda(1+\lambda) \\ 0 & 1-\lambda & 0 & \lambda \\ 0 & 0 & 1 & \lambda(1+\lambda) \end{array}\right).$$

To go further, we would have to divide the second row by $1-\lambda$. This can only be done if $1-\lambda \neq 0$, that is, if $\lambda \neq 1$, so let's assume this for the moment. If $\lambda \neq 1$:

$$\xrightarrow{\frac{1}{1-\lambda}R_2} \left(\begin{array}{ccc|c} 1 & \lambda & 0 & 1-\lambda(1+\lambda) \\ 0 & 1 & 0 & \frac{\lambda}{1-\lambda} \\ 0 & 0 & 1 & \lambda(1+\lambda) \end{array}\right) \xrightarrow{R_1-\lambda R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1-\lambda(1+\lambda)-\frac{\lambda^2}{1-\lambda} \\ 0 & 1 & 0 & \frac{\lambda}{1-\lambda} \\ 0 & 0 & 1 & \lambda(1+\lambda) \end{array}\right).$$

Since the left side in the RREF is an identity matrix, the system has one solution in this case.

We now need to consider the case $\lambda = 1$: After the second step above, we then have the matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

This is an inconsistent system, which has no solutions.

Conclusion: For $\lambda \neq 1$ there is a unique solution, namely

$$\begin{aligned} x &= 1 - \lambda(1+\lambda) - \frac{\lambda^2}{1-\lambda} \\ y &= \frac{\lambda}{1-\lambda} \\ z &= \lambda(1+\lambda). \end{aligned}$$

For $\lambda = 1$ there are no solutions, and for no value of $\lambda$ does the system have infinitely many solutions.

Note: in the second step above, we note that $(1-\lambda^2) = (1-\lambda)(1+\lambda)$. Instead of steps 2 and 3, we could have divided the row

$$\left( \begin{array}{ccc|c} 0 & 1-\lambda & 0 & \lambda \end{array} \right)$$

by $1-\lambda$ in order to get a 1 after the first 0. However, then we would have to assume that $\lambda \neq 1$, because we can't divide by 0. We would therefore have to consider two cases: first $\lambda \neq 1$ and then $\lambda = 1$. This is a perfectly fine approach, but here we tried to avoid splitting into cases for as long as we could.

**Example 14.3.** Find the values of $\lambda \in \mathbb{R}$ such that the following system has no solutions, one solution, or infinitely many solutions:

$$\left\{ \begin{array}{rcr} x+y+z &=& -1 \\ \lambda x+y+z &=& -1 \\ x+\lambda^2 y+z &=& \lambda \end{array} \right. .$$

Again, we use ERO to simplify the augmented matrix of the system:

$$\left( \begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ \lambda & 1 & 1 & -1 \\ 1 & \lambda^2 & 1 & \lambda \end{array} \right) \xrightarrow[R_3-R_1]{R_2-\lambda R_1} \left( \begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1-\lambda & 1-\lambda & \lambda-1 \\ 0 & \lambda^2-1 & 0 & \lambda+1 \end{array} \right)$$

If $\lambda = 1$, this results in $\left( \begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array} \right)$, which has no solution.

Otherwise, we continue:

$$\xrightarrow{R_2/(1-\lambda)} \left( \begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & \lambda^2-1 & 0 & \lambda+1 \end{array} \right) \xrightarrow{R_1-R_2} \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & \lambda^2-1 & 0 & \lambda+1 \end{array} \right)$$

Now, if $\lambda = -1$, this results in $\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right)$ which has infinitely many solutions.

Otherwise, we continue:

$$\xrightarrow{R_3/(\lambda+1)} \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & \lambda-1 & 0 & 1 \end{array} \right) \xrightarrow[R_3/(\lambda-1)]{R_2-R_3/(\lambda-1)} \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1-1/(\lambda-1) \\ 0 & 1 & 0 & 1/(\lambda-1) \end{array} \right)$$

which has a unique solutions.

Summarizing, the system has no solutions if $\lambda = 1$, infinitely many solutions if $\lambda = -1$, and a unique solution otherwise.

# Lecture 15

## The determinant

The determinant is an important number associated to a square matrix denoted by $\det A$ or $|A|$. We will define the determinant recursively. First, for a $1 \times 1$ matrix $A = (a)$ we set the determinant $\det A = a$. For an $n \times n$ matrix, we will now define the determinant in terms of determinants of $(n-1) \times (n-1)$ matrices.

*i)* Let $A = (a_{ij})$, for $1 \leq i, j \leq n$.

*ii)* For each $i$ and $j$, let $M_{ij}$ be the determinant of the matrix obtained by removing the $i$th row and the $j$th column of $A$.

*iii)* The *determinant* of $A$, denoted $\det A$ or $|A|$, is

$$\begin{aligned}\det A &= a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} + \cdots + (-1)^{n+1}a_{n1}M_{n1} \\ &= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} M_{i1}.\end{aligned}$$

The numbers $C_{ij} = (-1)^{i+j}M_{ij}$ are called *cofactors* of matrix $A$. The definition can be written in the form

$$\det A = \sum_{i=1}^{n} a_{i1}C_{i1},$$

this expression is called the *expansion of* $\det A$ *along the first column.*

**Example 15.1.** Compute $\det A$, where $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$. We have

$$M_{11} = \det(5) = 5, \qquad M_{21} = \det(2) = 2.$$

Thus $\det A = 1 \cdot 5 - 2 \cdot 2 = 1$.
In general,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

**Fact.** We can compute $\det A$ using expansion along *any* column: given $j = 1, \ldots, n$, we have

$$\det A = \sum_{i=1}^{n} a_{ij}C_{ij},$$

Moreover, we can also use *expansions along rows*: given $i = 1, \ldots, n$, we have

$$\det A = \sum_{j=1}^{n} a_{ij}C_{ij},$$

**Example 15.2.** Compute $\det A$, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$. Using expansion along the third row, we get

$$\det A = 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -2(-2) + 3 = 7.$$

If we expand along the second column, we obtain

$$\det A = -0 \cdot \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 3 - 2(-2) = 7.$$

**Properties of the determinant**

0) Determinant does not depend on the choice of the row or column in the expansion.

1) $\det I_n = 1$.

2) $\det A^T = \det A$.

3) Multiplicative: if $A$ and $B$ are two square matrices, we have

$$\det(AB) = \det A \cdot \det B.$$

4) If $A \in \mathrm{Mat}_n$ is diagonal, then $\det A = \prod_{i=1}^{n} a_{ii}$.

5) A matrix $A$ is called *upper-triangular* if $a_{ij} = 0$ for all $i > j$. For upper-triangular matrices $\det A = \prod_{i=1}^{n} a_{ii}$ as well.

Note that we know how to compute the determinant of an upper-triangular matrix, and we also know how to transform a matrix to an upper-triangular form by row operations. We are now interested in the following question: what is the behavior of the determinant under ERO?

# Lecture 16

### Matrices of ERO

ERO can be written in terms of multiplication of matrices. Denote by $I_{ij} \in \mathrm{Mat}_n$ the $n \times n$ matrix with $(ij)$-element equal to 1 and all other elements being zero. Let $A \in \mathrm{Mat}_n$. Then:

- ERO of type 1, i.e. adding row $k$ of $A$ multiplied by $\lambda$ to row $i$ of $A$, is equivalent to multiplying $A$ by $I + \lambda I_{ik}$ from the left:
$$A \xrightarrow{R_i + \lambda R_k} (I + \lambda I_{ik})A$$
Note that $\det(I + \lambda I_{ik}) = 1$, so by the multiplicative property of the determinant EROs of first type leave the determinant intact.

- ERO of type 2, i.e. swapping rows $i$ and $k$ of $A$, is equivalent to multiplying $A$ by $I - I_{ii} - I_{kk} + I_{ik} + I_{ki}$ from the left:
$$A \xrightarrow{R_i \leftrightarrow R_k} (I - I_{ii} - I_{kk} + I_{ik} + I_{ki})A$$
Since $\det(I - I_{ii} - I_{kk} + I_{ik} + I_{ki}) = -1$, EROs of second type change the sign of the determinant.

- ERO of type 3, i.e. multiplying row $i$ of $A$ by $\lambda$, is equivalent to multiplying $A$ by $I + (\lambda - 1)I_{ii}$ from the left:
$$A \xrightarrow{\lambda R_i} (I + (\lambda - 1)I_{ii})A$$
Since $\det(I + (\lambda - 1)I_{ii}) = \lambda$, EROs of third type multiply the determinant by $\lambda$.

Therefore, we can compute the determinant by Gauss elimination. Note that we can always transform a matrix to an upper-triangular form without using ERO of type 3 (though sometimes we may use them for convenience).

**Example 16.1.** Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -4 & -5 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{vmatrix} = -(-5) = 5.$$

**Example 16.2.** Let $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & 5 \\ -2 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{pmatrix}$. Then

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & 5 \\ -2 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 2 \\ 0 & 4 & 5 & 4 \\ 0 & 1 & -2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 4 & 5 & 4 \\ 0 & -5 & -1 & 2 \end{vmatrix} =$$

$$-\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & -11 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 7 \end{vmatrix} = -13 \cdot 7 = -91$$

**Algorithm**

Use Gauss elimination to transform a matrix $A$ to an upper-triangular matrix $T$ by ERO of types 1 and 2. Then $\det A = (-1)^m \det T$, where $m$ is the number of ERO of type 2 we have applied.

**Further properties of determinant**

6) If $A$ contains a zero row (or column) then $\det A = 0$.

7) If $A$ contains two similar rows (or columns) then $\det A = 0$.

## The inverse of a matrix

Let $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{R})$ be an $n \times n$ matrix, and $I = I_n$ the identity matrix (of size $n$). Then

$$AI = IA = A.$$

In other words, the identity matrix does not change another matrix, when multiplied.

Now, let $A\underline{x} = \underline{b}$ be a linear system and suppose that there is a matrix $B$ such that

$$BA = I.$$

Then we can multiply by $B$ on both sides:

$$BA\underline{x} = B\underline{b} \Longleftrightarrow \underline{x} = B\underline{b}.$$

This says that there exists a unique solution $\underline{x} = B\underline{b}$ to the system.

**Definition 16.3.** Let $A$ be a square matrix. A matrix $B$ is called the *inverse* of $A$ if

$$AB = BA = I.$$

The inverse of $A$ (if it exists) is denoted $A^{-1}$.

## Lecture 17

**Example 17.1.** The system

$$\begin{cases} x + 2y &= 1 \\ 2x + 5y &= -3 \end{cases}.$$

can be written $A\underline{x} = \underline{b}$, where $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Let

$$B = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}.$$

We can check that $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. Thus the solution is

$$\underline{x} = B\underline{b} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \end{pmatrix}.$$

So, we see that $B$ is the key to solving the system. Let's now compute the RREF:

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 5 & -3 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -5 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & -5 \end{array}\right) = (I \mid B\underline{b}).$$

In particular, the RREF is an identity matrix followed by a solution vector.

We saw above that the inverse of $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ is $\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$. The inverse does not always exist. For example, there is no inverse to the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

A matrix $A$ having an inverse is called *invertible.*

Observe that the matrix above has zero determinant. Determinant and inverse are related by the following statement:

**Theorem 17.2.** *A matrix $A$ has an inverse if and only if* $\det A \neq 0$.

Why is this theorem true? Assume first that $A$ has an inverse $A^{-1}$. Then, by the multiplicative property of the determinant,

$$\det A \det A^{-1} = \det(A \cdot A^{-1}) = \det I = 1,$$

which implies that $\det A \neq 0$.

Now we need to understand why does $\det A \neq 0$ imply the existence of the inverse. First, we note the following.

**Proposition 17.3.** *If $\det A \neq 0$ then a system of linear equations $A\underline{x} = \underline{b}$ has a unique solution. In particular, RREF of $A$ is the identity matrix.*

Indeed, observe that RREF of $A$ is obtained from $A$ by ERO, which implies that the determinant of the RREF is a non-zero multiple of $\det A$. Thus, if $\det A \neq 0$, the determinant of RREF is not zero either, so RREF of $A$ does not contain zero rows. Since it is a square matrix, by the definition of RREF we conclude that it is the identity matrix $I$, and the system has a unique solution.

Now, for every matrix $A$ with RREF of $A$ being the identity matrix we will explicitly construct the inverse.

**Algorithm: finding the inverse using ERO**

Let $\det A \neq 0$, so that the RREF of $A$ is the identity matrix $I_n$. Then we can transform $A$ to $I_n$ by ERO, let $k$ be the number of ERO required. Denote by $E_1, \dots, E_k$ the matrices of these ERO. Then

$$E_k E_{k-1} \dots E_1 \cdot A = I_n.$$

Denote $B = E_k E_{k-1} \dots E_1$, then we see that $BA = I_n$, so $B = A^{-1}$ is the required inverse of $A$! Further, if we create an augmented matrix $(A \,|\, I_n)$ and apply to it the row operations above, then we obtain

$$(A \,|\, I_n) \to (E_k E_{k-1} \dots E_1 \cdot A \,|\, E_k E_{k-1} \dots E_1 \cdot I_n) = (I_n \,|\, A^{-1} I_n) = (I_n \,|\, A^{-1}),$$

which leads to the following

**Algorithm.** Create the augmented matrix $(A \,|\, I_n)$, apply to it ERO to transform $A$ to $I_n$. Then the resulting matrix on the right is $A^{-1}$.

We can summarize the discussion above:

**Corollary 17.4.** *For $A \in \mathrm{Mat}_n$ the following are equivalent:*

- $\det A \neq 0$*;*
- *RREF of $A$ is the identity matrix $I_n$;*
- *$A$ is invertible.*

## Computing the inverse

Suppose that $A$ is a matrix which has an inverse (i.e., $\det A \neq 0$). To compute the inverse, we use the algorithm above based on the Gauss elimination on an augmented matrix.

**Example 17.5.** Compute the inverse of $A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$. We perform Gauss elimination on the augmented matrix

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1/3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_1}$$

$$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 1/3 & -2/3 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 1/3 & -2/3 & 1 & 0 \\ 0 & 0 & 1/3 & 4/3 & -2 & 1 \end{array}\right) \xrightarrow[R_2 - R_3]{R_1 - R_3}$$

$$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1/3 & 4/3 & -2 & 1 \end{array}\right) \xrightarrow{3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 4 & -6 & 3 \end{array}\right).$$

Once we reach an identity matrix to the left, we stop. The inverse of $A$ is on the right of the vertical line.

We now verify the answer:

$$\begin{pmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ 4 & -6 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Example 17.6.** Compute the inverse of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. As in the previous example, we perform Gauss elimination on the augmented matrix

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_2}$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right).$$

The matrix on the left has a zero row, so it cannot be transformed to the identity matrix. Therefore, $A$ is not invertible.

### Properties of the inverse

- $(A^{-1})^{-1} = A$ (since $A \cdot A^{-1} = I_n$).
- $(A^T)^{-1} = (A^{-1})^T$ (since $A^T \cdot (A^{-1})^T = (A \cdot A^{-1})^T = I_n$).
- $(AB)^{-1} = B^{-1}A^{-1}$ (since $B^{-1}A^{-1}AB = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$).
- If $A^{-1}$ exists then it is unique.

## Cofactor method

Recall that a *cofactor* $C_{ij}$ of a matrix $A \in \mathrm{Mat}_n$ is defined by $C_{ij} = (-1)^{i+j}M_{ij}$, where $M_{ij}$ is the determinant of an $(n-1) \times (n-1)$-matrix obtained from $A$ by removing $i$-th row and $j$-th column. Denote by $C$ the matrix composed of cofactors of $A$, and consider its transpose $C^T$, i.e. $(C^T)_{ij} = C_{ij}$.

**Example 17.7.** If $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, then $C = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$, so $C^T = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$. Thus,

$$AC^T = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_2 = (\det A)\, I_2.$$

In particular, this implies that $A \cdot \frac{1}{\det A}C^T = I_2$, and thus $A^{-1} = \frac{1}{\det A}C^T = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$.

## Lecture 18

Let us now follow Example 17.7 and compute $AC^T$.

$$(AC^T)_{ii} = \sum_{k=1}^{n} a_{ik}(C^T)_{ki} = \sum_{k=1}^{n} a_{ik}C_{ik} = \det A,$$

so all the diagonal elements of the matrix $AC^T$ are equal to $\det A$. Now, we need to compute

$$(AC^T)_{ij} = \sum_{k=1}^{n} a_{ik}(C^T)_{kj} = \sum_{k=1}^{n} a_{ik}C_{jk}.$$

Given $i \neq j$, create a new matrix $A'$ in the following way: take matrix $A$ and substitute $j$-th row by $i$-th row, i.e.

$$(A')_{lk} = \begin{cases} a_{lk} & \text{if } l \neq j \\ a_{ik} & \text{if } l = j \end{cases}$$

Observe that $A'$ has two similar rows ($i$-th and $j$-th ones), so its determinant is equal to zero. Therefore, for $i \neq j$

$$(AC^T)_{ij} = \sum_{k=1}^{n} a_{ik}(C^T)_{kj} = \sum_{k=1}^{n} a_{ik}C_{jk} = \sum_{k=1}^{n} (A')_{jk}C_{jk} = \det A' = 0.$$

Thus, $AC^T = (\det A) \cdot I_n$, so $A\left(\frac{1}{\det A}C^T\right) = I_n$, which implies

**Corollary 18.1.**

$$A^{-1} = \frac{1}{\det A}C^T.$$

**Example 18.2.** Let $A \in \mathrm{Mat}_2$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, $C^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $\det A = ad - bc$, so

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For example, for $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ we have

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}.$$

### Special types of matrices

We will list several important types of (square) matrices that show up frequently in various applications.

**Diagonal and triangular matrices.** Recall that a square matrix $A = (a_{ij})$ is **diagonal** if $a_{ij} = 0$ for all $i \neq j$, i.e. all off-diagonal elements are equal to zero. For example, $A = \begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix}$ is diagonal. Some easy properties of diagonal matrices:

- $\det A = \Pi_{i=1}^{n} a_{ii}$;
- $A^{-1}$ (if exists) is also diagonal with $(A^{-1})_{ii} = a_{ii}^{-1}$;
- any two diagonal matrices $A, B \in \mathrm{Mat}_n$ *commute*, i.e. $AB = BA$.

Recall also that a square matrix $A$ is called **upper-triangular** if $a_{ij} = 0$ for $i > j$. Similarly, we can define a **lower-triangular** matrix as one with $a_{ij} = 0$ for $i < j$. Some properties:

- If $A$ is upper- or lower-triangular, then $\det A = \Pi_{i=1}^{n} a_{ii}$;
- $A^{-1}$ of an upper-triangular (lower-triangular) matrix (if exists) is also upper-triangular (respectively, lower-triangular);
- a product of two upper-triangular (lower-triangular) matrices is also upper-triangular (respectively, lower-triangular).

**Symmetric, anti-symmetric, Hermitian and anti-Hermitian matrices.** Recall that a matrix $A$ is **symmetric** if $A^T = A$, i.e. $a_{ij} = a_{ji}$ for all $i, j$. We say that a matrix is **anti-symmetric** (or **skew-symmetric**) if $A^T = -A$, e.g. $A = \begin{pmatrix} -4 & -2 \\ 2 & -1 \end{pmatrix}$ is skew-symmetric. Note that every square matrix can be written as a sum of a symmetric and a skew-symmetric ones:

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}.$$

Complex analogs of real symmetric matrices are Hermitian matrices. For a complex number $z$ we denote by $z^*$ the complex conjugate of $z$ (note that in mathematics it is usually denoted by $\bar{z}$), and by $A^*$ the matrix with all entries being conjugate, i.e. $(A^*)_{ij} = a_{ij}^*$. A **Hermitian conjugate** of a matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ is the matrix $A^\dagger = (A^T)^*$.

**Example 18.3.** Let $A = \begin{pmatrix} -4i & 2+i \\ 3i & -1 \end{pmatrix}$. Then $A^T = \begin{pmatrix} -4i & 3i \\ 2+i & -1 \end{pmatrix}$, so the Hermitian conjugate of $A$ is $A^\dagger = \begin{pmatrix} 4i & -3i \\ 2-i & -1 \end{pmatrix}$.

A matrix $A$ is called **Hermitian** if $A^\dagger = A$, and **anti-Hermitian** if $A^\dagger = -A$.

**Example 18.4.** A matrix $A = \begin{pmatrix} -4 & 2+i \\ 2-i & -1 \end{pmatrix}$ is Hermitian, and $B = \begin{pmatrix} -4i & -2+i \\ 2+i & 0 \end{pmatrix}$ is anti-Hermitian.

Note that any symmetric matrix with real values is Hermitian.

**Orthogonal matrices.** A matrix $A \in \mathrm{Mat}_n$ is called **orthogonal** if $AA^T = I_n$, i.e. $A^{-1} = A$. Since $\det A^{-1} = (\det A)^{-1}$, we see that the square of the determinant of any orthogonal matrix is equal to 1:

$$1 = \det I = \det(A^{-1}A) = \det A \det A^{-1} = (\det A)^2,$$

so $\det A = \pm 1$.

**Example 18.5.** Matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and, in general $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ are orthogonal.

The set of all orthogonal $(n \times n)$ matrices is denoted by $O(n)$. The main property of othogonal matrices is the following. For any vector $\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ define its length by $\sqrt{x^2 + y^2}$ (cf. Pythagoras Theorem). Then multiplication by an orthogonal matrix does not change the length.

**Example 18.6.**

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\varphi - y\sin\varphi \\ x\sin\varphi + y\cos\varphi \end{pmatrix},$$

and

$$(x\cos\varphi - y\sin\varphi)^2 + (x\sin\varphi + y\cos\varphi)^2 = x^2 + y^2.$$

This property holds for an orthogonal matrix of any size. Also, since $(A^T)^{-1} = (A^{-1})^T$, an inverse of an orthogonal matrix is also orthogonal. Further, a product of orthogonal matrices is also an orthogonal: if $A, B \in O(n)$, then

$$(AB)^T(AB) = B^T A^T AB = B^T(A^T A)B = B^T IB = B^T B = I.$$

## Lecture 19

**Unitary and normal matrices** A complex matrix $A$ is called **unitary** if $A^\dagger A = I$, i.e. $A^\dagger = A^{-1}$. Note that real orthogonal matrices are also unitary. The set of all unitary $(n \times n)$ matrices is denoted by $U(n)$.

**Example 19.1.** A matrix $A = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$ is unitary: $A^\dagger = A^* = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$, so $A^\dagger A = I$.

Similarly to orthogonal matrices, an inverse of a unitary matrix is also unitary, and a product of unitary matrices is a unitary matrix. Also,

$$1 = \det I = \det(A^{-1}A) = \det(A^\dagger A) = \det A^\dagger \det A = (\det A)^* \det A = |\det A|^2,$$

so $|\det A| = 1$.

Unitary matrices preserve the length $l(\underline{v})$ of a complex vector $\underline{v} = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix}$ defined by $l(\underline{v}) = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$.

A matrix $A$ is called **normal** if $A^\dagger A = AA^\dagger$, i.e. if it commutes with its Hermitian conjugate. For example, Hermitian and unitary matrices are normal. An inverse of a normal matrix (if exists) is also normal.

### Vector spaces

Let

$$\mathbb{R}^n = \{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_i \in \mathbb{R} \}$$

be the set of $n \times 1$ column vectors of real numbers. Similarly, if we replace $\mathbb{R}$ by $\mathbb{C}$ we get $\mathbb{C}^n$. We can add two vectors

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

and we can multiply a vector by a scalar $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$ if we work over $\mathbb{C}$):

$$\lambda \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$

(but note that we cannot multiply two vectors because the matrix product is not defined for two $n \times 1$ matrices, unless $n = 1$!)

**Definition 19.2.** A vector space is a set with two operations: addition and scalar multiplication. Its elements are called *vectors*. In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces.

**Definition 19.3.** Let $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m$ be vectors and $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ (or $\mathbb{C}$). The vector

$$\lambda_1\underline{v}_1 + \lambda_2\underline{v}_2 + \cdots + \lambda_m\underline{v}_m$$

is called a *linear combination* of $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m$.

**Example 19.4.** Let $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$. Then any vector $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ is a linear combination of $\underline{v}_1$ and $\underline{v}_2$ because

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x\underline{v}_1 + y\underline{v}_2.$$

Note that the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is *not* a linear combination of $\underline{v}_1$ and $\underline{v}_2$.

A set of vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m$ is called *linearly dependent* if one of the vectors is a linear combination of the others, that is, if

$$\underline{v}_i = \lambda_1\underline{v}_1 + \cdots + \lambda_{i-1}\underline{v}_{i-1} + \lambda_{i+1}\underline{v}_{i+1} + \cdots + \lambda_m\underline{v}_m,$$

for some $1 \leq i \leq m$. This is equivalent to saying that there exist scalars $\lambda_1, \dots, \lambda_m$ (not all of them zero!) such that

$$\lambda_1\underline{v}_1 + \cdots + \lambda_m\underline{v}_m = \underline{0}.$$

If $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m$ are not linearly dependent, they are said to be *linearly independent.* Mathematically, this means that the relation

$$\lambda_1\underline{v}_1 + \cdots + \lambda_m\underline{v}_m = \underline{0}$$

can only hold if $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$.

**Example 19.5.**

- $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$ are linearly dependent, because

  $$2\underline{v}_1 - \underline{v}_2 = \underline{0}.$$

- $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent, because if

  $$\lambda_1\underline{v}_1 + \lambda_2\underline{v}_2 = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ -\lambda_2 \end{pmatrix} = \underline{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  then we must have

  $$\lambda_1 + \lambda_2 = 0, \quad \text{and} \quad -\lambda_2 = 0,$$

  that is, $\lambda_1 = \lambda_2 = 0$.

- $\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$ are linearly dependent because we have the relation

$$\underline{v}_1 + \underline{v}_2 = \underline{v}_3.$$

**Definition 19.6.** The *span* of the vectors $\underline{v}_1, \ldots, \underline{v}_m$, written

$$\text{span}\{\underline{v}_1, \ldots, \underline{v}_m\},$$

is the set of all vectors which are linear combinations of $\underline{v}_1, \ldots, \underline{v}_m$.

**Example 19.7.**

- $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = \mathbb{R}^2$, since for any $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ there are $\lambda_1, \lambda_2$ such that $\begin{pmatrix} a \\ b \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, namely,$\lambda_2 = -b, \quad \lambda_1 = a + b$.

- $$\begin{aligned} \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} &= \left\{\lambda_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \\ &= \left\{(\lambda_1 + \lambda_3) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (\lambda_2 + \lambda_3) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}. \end{aligned}$$

# Lecture 20

## Bases

Let $V$ be a vector space (e.g., $\mathbb{R}^n$ or $\mathbb{C}^n$). The span of some vectors in $V$ is also a vector space, which can be all of $V$ or smaller.

**Definition 20.1.** A *basis* of $V$ is a set of vectors $\underline{v}_1, \ldots, \underline{v}_m$ such that:

*i)* this set is linearly independent,

*ii)* $\operatorname{span}\{\underline{v}_1, \ldots, \underline{v}_m\} = V$.

**Example 20.2.** The vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

form a basis for $\mathbb{R}^2$ (or $\mathbb{C}^2$), called the *standard basis*:

- They span all of $\mathbb{R}^2$ (or $\mathbb{C}^2$): any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can be written as

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- They are linearly independent: If $\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{0}$, then $\lambda_1 = \lambda_2 = 0$.

There are other bases for $\mathbb{R}^2$, for example

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

from Example (19.7).

On the other hand, the three vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

also span $\mathbb{R}^2$, but they are *not* linearly independent:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so these three vector do not form a basis.

**Theorem 20.3.** *Every vector space has a basis. For a given vector space $V$, the number of elements in a basis (if finite) is always the same. This number is called the* dimension *of $V$ (notation:* $\dim V$*).*

For example, $\mathbb{R}^2$ (or $\mathbb{C}^2$) has dimension two. In $\mathbb{R}^3$ (or $\mathbb{C}^3$) we have the standard basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

so these spaces have dimension 3. More generally, the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are $n$-dimensional.

## Matrices as linear maps

Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be a square matrix of size $n$. We can define a function from the vector space $\mathbb{R}^n$ to itself:

$$L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad L_A(\underline{x}) = A\underline{x}.$$

This map is compatible with addition and scalar multiplication, that is,

$$L_A(\underline{x} + \underline{y}) = A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y} = L_A(\underline{x}) + L_A(\underline{y})$$

and

$$L_A(\lambda\underline{x}) = A(\lambda\underline{x}) = \lambda A\underline{x} = \lambda L_A(\underline{x}), \qquad \lambda \in \mathbb{R}.$$

A function satisfying these two properties is called a *linear map.*

So, from a matrix, we get a linear map. We can also go the other way:

> Given a linear map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, we can write down a matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, such that $L_A = f$.

The way to do this is the following. Let $\{\underline{v}_1, \ldots, \underline{v}_n\}$ be the standard basis of $\mathbb{R}^n$, and $\{\underline{u}_1, \ldots, \underline{u}_m\}$ is the standard basis in $\mathbb{R}^m$. Then every $f(\underline{v}_j)$ is a linear combination of vectors of $\{\underline{u}_1, \ldots, \underline{u}_m\}$, so we can write for every $j = 1, \ldots, n$

$$f(\underline{v}_j) = a_{1j}\underline{u}_1 + a_{2j}\underline{u}_2 + \cdots + a_{mj}\underline{u}_m = \sum_{i=1}^{m} a_{ij}\underline{u}_i.$$

Then $f = L_A$, where $A = (a_{ij})$.

Here is an example to show how this is done.

**Example 20.4.** Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the function

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -y \\ x \end{pmatrix}.$$

We first show that $f$ is a linear map:

- Additivity:
$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = f\begin{pmatrix} x+a \\ y+b \\ z+c \end{pmatrix} = \begin{pmatrix} z+c \\ -(y+b) \\ x+a \end{pmatrix} = \begin{pmatrix} z \\ -y \\ x \end{pmatrix} + \begin{pmatrix} c \\ -b \\ a \end{pmatrix} = f\begin{pmatrix} x \\ y \\ z \end{pmatrix} + f\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$
- Scalar multiplication: $f\left(\lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \lambda z \\ -\lambda y \\ \lambda x \end{pmatrix} = \lambda f\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$

We will now find a matrix $A$ such that $L_A = f$. To do this, we choose the standard basis of $\mathbb{R}^n$.:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

To find the matrix with respect to this basis, we evaluate $f$ on each basis vector:

$$\begin{aligned} f\begin{pmatrix}1\\0\\0\end{pmatrix} &= \begin{pmatrix}0\\0\\1\end{pmatrix} \\ f\begin{pmatrix}0\\1\\0\end{pmatrix} &= \begin{pmatrix}0\\-1\\0\end{pmatrix} \\ f\begin{pmatrix}0\\0\\1\end{pmatrix} &= \begin{pmatrix}1\\0\\0\end{pmatrix}. \end{aligned}$$

We then put the three resulting vectors together as the columns of a matrix:

$$A = \begin{pmatrix}0 & 0 & 1\\0 & -1 & 0\\1 & 0 & 0\end{pmatrix}.$$

Then our original map $f$ equals $L_A$:

$$L_A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\0 & -1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\-y\\x\end{pmatrix} = f\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

# Lecture 21

### Kernel, image and rank

Let $A \in \mathrm{Mat}_{m\times n}$, and $L_A : V \to W$ be a linear map. Here $\dim V = n$, $\dim W = m$.

**Definition 21.1.**
- A *kernel* of $L_A$ is the set $\ker L_A = \{\underline{v} \in V \mid L_A(\underline{v}) = \underline{0}\}$.
- An *image* of $L_A$ is the set $\operatorname{im} A = \{\underline{w} \in W \mid \underline{w} = L_A(\underline{v}) \text{ for some } \underline{v} \in V\}$.
- Recall that a kernel of a matrix $A$ is the set of solutions of the homogeneous system $A\underline{x} = \underline{0}$. Thus, the kernel of $A$ is precisely the kernel of $L_A$.

**Example 21.2.**
- $L_A = 0 : V \to W$, $\underline{v} \mapsto \underline{0} \in W$. Then $\ker L_A = V$, $\operatorname{im} L_A = \underline{0} \in W$.
- $L_A = \mathrm{id} : V \to V$, $\underline{v} \mapsto \underline{v}$ (*identity map*). Then $\ker L_A = \underline{0}$, $\operatorname{im} L_A = V$.
- $L_A : \mathbb{R}^2 \to \mathbb{R}$, $L_A \begin{pmatrix} x \\ y \end{pmatrix} = x$. Then $\ker L_A = \{\begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{R}\}$, $\operatorname{im} L_A = \mathbb{R}$.

Note that both the image and the kernel of a linear map are vector spaces themselves.

**Definition 21.3.** A *rank* of a linear map $L_A$ is the dimension of its image. Equivalently, it is the maximal number of linearly independent columns in the corresponding matrix $A$. The latter is called a *rank* of $A$, notation: $\operatorname{rk} A$.

**Example 21.4.** $\operatorname{rk}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$; $\operatorname{rk}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1$; $\operatorname{rk}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$; $\operatorname{rk}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2$.

**Fact.** Let $A \in \mathrm{Mat}_{m\times n}$. Then
- $\operatorname{rk} A$ is also equal to the maximal number of linearly independent *rows* of $A$;
- therefore, $\operatorname{rk} A$ does not change under ERO;
- thus, the rank of $A$ is actually equal to the number of non-zero rows in the RREF of $A$.

**Example 21.5.** Let $A = \begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 2 & 1 & 6 \end{pmatrix}$. The columns of $A$ are the vectors $\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{v}_4 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$. Vectors $\underline{v}_1$ and $\underline{v}_2$ are linearly independent, but the other vectors are linear combinations of $\underline{v}_1$ and $\underline{v}_2$: $\underline{v}_3 = \underline{v}_2 - \underline{v}_1$, $\underline{v}_4 = 2\underline{v}_2$. Therefore, the maximal number of linear independent columns is 2, so $\operatorname{rk} A = 2$.

On the other hand, we can compute RREF of $A$:

$$\begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 2 & 1 & 6 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 1 & 6 \\ 0 & 1 & 1 & 3 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

Thus, the RREF of $A$ has two non-zero rows, so we see again that $\operatorname{rk} A = 2$.

Let now $A \in \mathrm{Mat}_n$. We say that $A$ is *singular* if $\operatorname{rk} A < n$, and *non-singular* otherwise. As we can see from the fact above, the rank of a square matrix $A$ is equal to $n$ if and only if the RREF of $A$ is the identity matrix $I_n$, which, as we know (see Proposition 17.3), is the same as $\det A \neq 0$. Therefore, in view of Corollary 17.4, we can summarize this as follows:

**Corollary 21.6.** *For $A \in \mathrm{Mat}_n$ the following are equivalent:*

- $\operatorname{rk} A = n$ *(i.e., $A$ is non-singular);*
- $\ker A = \underline{0}$ *(i.e. the homogeneous system $A\underline{x} = \underline{0}$ has a unique solution);*
- $\det A \neq 0$*;*
- *RREF of $A$ is the identity matrix $I_n$;*
- *$A$ is invertible.*

There is another way to find the rank of a matrix $A \in \mathrm{Mat}_{m\times n}$ based on the following property.

**Fact.** The rank of a matrix is equal to the maximal size of a non-singular square submatrix.

**Example 21.7.** Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Since there are two rows only, we see that $\operatorname{rk} A \leq 2$. We observe that $\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$, so there is a $2 \times 2$ non-singular submatrix, which implies that $\operatorname{rk} A \geq 2$. Therefore, $\operatorname{rk} A = 2$.

**Example 21.8.** Let $A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & -4 \\ 0 & -3 & 6 \end{pmatrix}$. Observe that $\det A = 0$, so $\operatorname{rk} A < 3$. Further, any $2 \times 2$ submatrix of $A$ is singular, so $\operatorname{rk} A < 2$. There are non-zero entries in $A$, which implies that $\operatorname{rk} A \geq 1$. Thus, $\operatorname{rk} A = 1$.

# Lecture 22

**One more fact about the rank.**

**Definition 22.1.** Let $A \in \mathrm{Mat}_{m\times n}$. The dimension of the kernel of $A$ is called *nullity* of $A$, notation $\operatorname{null} A$.

Nullity is closely related to rank:

**Proposition 22.2.** *For $A \in \mathrm{Mat}_{m\times n}$, we have* $\operatorname{null} A = n - \operatorname{rk} A$.

The reason for this is the following: the kernel is the set of solutions of the homogeneous system $A\underline{x} = \underline{0}$, so the dimension of the kernel is equal to the number of "free parameters" in the solution of the system. The rank is equal to the number of non-zero rows in RREF, so it is equal to the number of "non-free" variables. Therefore, these two numbers sum up to the number of variables, i.e. to $n$.

## Application to linear ODEs

Ordinary Differential Equations (ODEs) come up in the modelling of engineering and physical problems. We can use matrices to help solve linear ODEs:

**Example 22.3.** Solve the ODE

$$y'' - 5y' + 4y = 0,$$

where $y = y(t)$ is a function in $t$, with the initial conditions $y(3) = 6$, $y'(3) = -1$.
Solution: We can write higher order ODE as a system with a change of variables. Let

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t). \end{aligned}$$

Taking derivatives, we get

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -4y + 5y' = -4x_1 + 5x_2. \end{aligned}$$

The initial conditions become

$$x_1(3) = 6, \qquad x_2(3) = -1.$$

Our ODE is thus rewritten as

$$\underline{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\underline{x},$$

where $A$ is the $2 \times 2$ matrix.

Now if we had a one-variable ODE $x' = ax$, for $a \in \mathbb{R}$, then the solution would be $x(t) = ce^{at}$, for some constant $c$. For our equation $\underline{x}' = A\underline{x}$, let's see when

$$\underline{x} = \underline{b}e^{rt}$$

is a solution, for some vector $\underline{b}$ and $r \in \mathbb{R}$. Well, this will be a solution precisely when

$$\underline{x}' = \underline{b} r e^{rt} = A\underline{b} e^{rt}.$$

Cancelling the $e^{rt}$ (which are never zero!), we get

$$A\underline{b} = r\underline{b}.$$

So, we need to find the vectors $\underline{b}$ satisfying this. Such vectors are called *eigenvectors* of $A$ with *eigenvalue* $r$.

To find these, we do the following:

- Compute the determinant of the matrix $A - \lambda I_2$:

$$\det(A - \lambda I_2) = \left| \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \left| \begin{pmatrix} -\lambda & 1 \\ -4 & 5-\lambda \end{pmatrix} \right| = -\lambda(5-\lambda) + 4$$
$$= \lambda^2 - 5\lambda + 4.$$

  Now find the roots of this polynomial:

$$\lambda_1 = 1, \quad \lambda_2 = 4.$$

  These are the eigenvalues of $A$.

- Next, solve the equation

$$A\underline{b} = r\underline{b}$$

  for each of the eigenvalues. For the first one:

$$\begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 1 \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \iff \begin{pmatrix} b_2 \\ -4b_1 + 5b_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
$$\iff \begin{cases} b_2 & = b_1 \\ -4b_1 + 5b_2 & = b_2 \end{cases} \iff b_2 = b_1.$$

  We only need one solution, for example

$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

  For the second eigenvalue, we similarly get a solution

$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

We now return to our system of ODEs: $\underline{x}' = A\underline{x}$, and see that we have found two solutions to it:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \qquad \underline{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t}.$$

We now finish by using the following fact:

**Fact.** The set of solutions $\underline{x}$ (which are functions in $t$) of the system $\underline{x}' = A\underline{x}$ form a vector space. In fact, this space equals

$$\text{span}\,\{\underline{v}_1, \underline{v}_2\}\,.$$

Thus, any solution of $\underline{x}' = A\underline{x}$ is a linear combination of $\underline{v}_1$ and $\underline{v}_2$, that is, the general solution is

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\underline{v}_1 + c_2\underline{v}_2,$$

for some $c_1, c_2 \in \mathbb{R}$.

Plugging in our initial values

$$x_1(3) = 6, \qquad x_2(3) = -1,$$

we can find the constants:

$$\begin{aligned} \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} &= \begin{pmatrix} 6 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^3 + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4\cdot 3}, \\ &\iff \begin{cases} c_1e^3 + c_2e^{12} &= 6 \\ c_1e^3 + 4c_2e^{12} &= -1 \end{cases}. \end{aligned}$$

Solving this, we get

$$c_1 = \frac{25}{3e^3}, \qquad c_2 = \frac{-7}{3e^{12}}.$$

So, the solution to our original equation is

$$y(t) = x_1(t) = c_1e^t + c_2e^{4t} = \frac{25}{3e^3}e^t + \frac{-7}{3e^{12}}e^{4t}.$$

The above example shows that it is of interest of find eigenvalues and eigenvectors of matrices.

**Example 22.4.** Let

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}.$$

Find the eigenvalues: $\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{pmatrix} = (3-\lambda)(-\lambda) + 2 = \lambda^2 - 3\lambda + 2$. Find the roots:

$$\lambda = 1, \qquad \lambda = 2.$$

Now find eigenvectors for each eigenvalue:

$$\begin{aligned} \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} = 0 \iff \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = 0 \\ \iff 2x + y = 0, \end{aligned}$$

so one eigenvector is, for example,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Note that eigenvectors are always defined up to scaling.
For the eigenvalue $\lambda = 2$ we similarly get an eigenvector

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus we have

$$\begin{aligned} A \begin{pmatrix} 1 \\ -2 \end{pmatrix} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ A \begin{pmatrix} 1 \\ -1 \end{pmatrix} &= 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}. \end{aligned}$$

That is, the linear map $L_A$ fixes one eigenvector, and doubles the other.

# Lecture 23

### Eigenvalues and eigenvectors

**Definition 23.1.** Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Recall that if

$$A\underline{x} = \lambda \underline{x},$$

for some *non-zero* vector $\underline{x} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, then $\lambda$ is called an *eigenvalue* of $A$ and $\underline{x}$ is called an *eigenvector* of $A$ (for the eigenvalue $\lambda$).

To find eigenvalues and eigenvectors, we proceed as follows:
Rewrite the equation $A\underline{x} = \lambda\underline{x}$ as

$$A\underline{x} - \lambda\underline{x} = (A - \lambda I)\underline{x} = \underline{0}.$$

We thus have a homogeneous system of linear equations, with coefficient matrix $A - \lambda I$. A linear system has either zero solutions, one solution or infinitely many solutions. A homogeneous system always has at least one solution: $\underline{x} = 0$, so the first possibility is excluded.

Now, if the determinant $\det(A - \lambda I)$ is not zero, then we know that $A - \lambda I$ has an inverse, so we would get *exactly one* solution

$$(A - \lambda I)^{-1}(A - \lambda I)\underline{x} = 0 \Longrightarrow \underline{x} = \underline{0}.$$

But an eigenvector is not allowed to be 0, so we will ignore this case. Thus, the only possibility is that

$$\det(A - \lambda I) = 0,$$

and for any $\lambda$ satisfying this, we will have infinitely many solutions $\underline{x}$.

The LHS here will be a polynomial in $\lambda$ of degree $n$; compare how in (22.3) we got

$$\det(A - \lambda I_2) = \lambda^2 - 5\lambda + 4.$$

The polynomial $\det(A - \lambda I)$ is called the *characteristic polynomial* of $A$. Its roots are those values of $\lambda$ for which the equation $A\underline{x} = \lambda\underline{x}$ has a non-zero solution, so these roots are the eigenvalues of $A$.

Suppose now that $a$ is an eigenvalue of $A$. To find the corresponding eigenvector(s), we solve the linear system

$$A\underline{x} = a\underline{x},$$

just like we did in (22.3). Note: We will have infinitely many eigenvectors.

We have seen examples at the last lecture, here are some other examples:

**Example 23.2.** Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The characteristic polynomial is $\det(A - \lambda I) = 1 + \lambda^2$, so the eigenvalues are $\pm i$. The eigenvectors $\underline{v}_1$ and $\underline{v}_2$ can be found by solving the equations $(A - iI)\underline{v}_1 = \underline{0}$ and $(A - iI)\underline{v}_2 = \underline{0}$, so we can choose eigenvectors $\underline{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

**Example 23.3.** Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The characteristic polynomial is $\det(A - \lambda I) = (-\lambda)^2 = \lambda^2$, so there is only *one* eigenvalue $\lambda = 0$. The space of eigenvectors $E_0$ is given by

$$E_0 = \{\underline{x} \in \mathbb{C}^2 \mid A\underline{x} = \underline{0}\},$$

that is,

$$E_0 = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{C}\}.$$

Since $E_0$ is spanned by one vector, for example $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, it means that $E_0$ is a one-dimensional vector space. This means that any two vectors is $E_0$ are linearly dependent. Thus, there are no two vectors in $E_0$ which form a basis for $\mathbb{C}^2$.

**Definition 23.4.** The set of eigenvectors for an eigenvalue $\lambda$ is called an *eigenspace*, denoted $E_\lambda$. Thus,

$$E_\lambda = \{\underline{v} \mid (A - \lambda I)\underline{v} = \underline{0}\}.$$

**Example 23.5.** Let $A = \begin{pmatrix} 1 & 0 & 6 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix}$. Determine the eigenvalues and eigenspaces of $A$.

Solution: For the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 6 \\ 3 & 2-\lambda & 1 \\ 2 & 0 & 2-\lambda \end{vmatrix} &= \text{(expand along the middle column)} \\ = (2-\lambda)\begin{vmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{vmatrix} &= (2-\lambda)((1-\lambda)(2-\lambda) - 12) \\ = (2-\lambda)(\lambda^2 - 3\lambda - 10) &= (2-\lambda)(\lambda - 5)(\lambda + 2). \end{aligned}$$

The last step is given by finding the roots of the quadratic polynomial. Thus the eigenvalues are

$$2, \quad 5, \quad -2.$$

The eigenvectors in the eigenspace $E_2$ are given by

$$A\underline{x} = 2\underline{x} \iff (A - 2I)\underline{x} = \underline{0} \iff \begin{pmatrix} -1 & 0 & 6 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} \underline{x} = 0.$$

Gauss elimination gives the equivalent system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = 0 \iff \underline{x} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, \text{ for any } y \in \mathbb{C}.$$

Thus

$$E_2 = \left\{ \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \mid y \in \mathbb{C} \right\}.$$

Similarly, for the eigenvalue 5, we get

$$\begin{pmatrix} -4 & 0 & 6 \\ 3 & -3 & 1 \\ 2 & 0 & -3 \end{pmatrix} \underline{x} = 0 \iff \begin{pmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -11/6 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = 0$$

$$\iff \underline{x} = \begin{pmatrix} 3z/2 \\ 11z/6 \\ z \end{pmatrix}, \ z \in \mathbb{C}.$$

$$\iff E_5 = \left\{ \begin{pmatrix} 3z/2 \\ 11z/6 \\ z \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

Finally, for $-2$, we get

$$\begin{pmatrix} 3 & 0 & 6 \\ 3 & 4 & 1 \\ 2 & 0 & 4 \end{pmatrix} \underline{x} = 0 \iff \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = 0$$

$$\iff \underline{x} = \begin{pmatrix} -2z \\ 5z/4 \\ z \end{pmatrix}, \ z \in \mathbb{C}.$$

$$\iff E_{-2} = \left\{ \begin{pmatrix} -2z \\ 5z/4 \\ z \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

We see that each of the eigenspaces are one-dimensional. Indeed, we can choose a one-element basis in each:

$$E_2 = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \qquad E_5 = \operatorname{span}\left\{ \begin{pmatrix} 9 \\ 11 \\ 6 \end{pmatrix} \right\}, \qquad E_{-2} = \operatorname{span}\left\{ \begin{pmatrix} 8 \\ -5 \\ -4 \end{pmatrix} \right\}.$$

**Example 23.6.** Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$. Determine the eigenvalues and eigenspaces of $A$.

Solution: For the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & -2 & -\lambda \end{vmatrix} = (1-\lambda)^2(4+\lambda^2) = (1-\lambda)^2(2i+\lambda)(2i-\lambda).$$

Thus the eigenvalues are

$$\lambda_1 = 2i, \ \lambda_2 = -2i, \lambda_3 = 1.$$

The eigenvectors in the eigenspace $E_{2i}$ are given by

$$\underline{0} = (A - 2iI)\underline{x} = \begin{pmatrix} 1-2i & 0 & 0 & 0 \\ 0 & 1-2i & 0 & 0 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{pmatrix} \underline{v}_1,$$

which is equivalent to

$$\underline{v}_1 = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix}, \; c_1 \in \mathbb{C}.$$

Thus

$$E_{2i} = \left\{ \begin{pmatrix} 0 \\ 0 \\ c_1 \\ ic_1 \end{pmatrix} \mid c_1 \in \mathbb{C} \right\}.$$

Similarly, for the eigenvalue $\lambda_2 = -2i$, we get

$$\underline{0} = (A - 2iI)\underline{x} = \begin{pmatrix} 1+2i & 0 & 0 & 0 \\ 0 & 1+2i & 0 & 0 \\ 0 & 0 & 2i & 2 \\ 0 & 0 & -2 & 2i \end{pmatrix} \underline{v}_2,$$

which is equivalent to

$$\underline{v}_2 = c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix}, \; c_2 \in \mathbb{C}.$$

Thus

$$E_{-2i} = \left\{ \begin{pmatrix} 0 \\ 0 \\ c_2 \\ -ic_2 \end{pmatrix} \mid c_2 \in \mathbb{C} \right\}.$$

Finally, for $\lambda_3 = 1$, we get

$$\underline{0} = (A - I)\underline{x} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix} \underline{v},$$

which is equivalent to

$$\underline{v} \in E_1 = \left\{ \begin{pmatrix} c_3 \\ c_4 \\ 0 \\ 0 \end{pmatrix} \mid c_3, c_4 \in \mathbb{C} \right\}.$$

We see that $E_1$ can be written as a span of the two linearly independent vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, so it is two-dimensional. Note also that we could choose a different basis in $E_1$, and thus obtain a different basis of $\mathbb{C}^4$ consisting of eigenvectors of $A$. We will come back to this in the process of diagonalization of matrices.

# Lecture 24

### Eigenvalues of special matrices

We now look at eigenvalues of matrices of special types.

**Hermitian matrices.** Let $A \in \mathrm{Mat}_n$ be Hermitian, so that $A^\dagger = A$. Let $\underline{v}$ be an eigenvector of $A$, $A\underline{v} = \lambda\underline{v}$. Let us compute $\underline{v}^\dagger A\underline{v}$ in two ways. First,

$$\underline{v}^\dagger A\underline{v} = (\underline{v}^\dagger A)\underline{v} = (\underline{v}^\dagger A^\dagger)\underline{v} = (A\underline{v})^\dagger\underline{v} = (\lambda\underline{v})^\dagger\underline{v} = \underline{v}^\dagger\lambda^*\underline{v} = \lambda^*(\underline{v}^\dagger\underline{v}).$$

On the other hand,

$$\underline{v}^\dagger A\underline{v} = \underline{v}^\dagger(A\underline{v}) = \underline{v}^\dagger\lambda\underline{v} = \lambda(\underline{v}^\dagger\underline{v}).$$

Comparing the two expressions and using the fact $\underline{v}^\dagger\underline{v} \neq 0$, we see that $\lambda^* = \lambda$. Therefore, eigenvalues of Hermitian matrices are always real.

**Example 24.1.** Let $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$. Then the characteristic polynomial is $\det(A - \lambda I) = (1 - \lambda)(2 - \lambda) - 1 = \lambda^2 - 3\lambda + 1$ which has two real roots $\lambda = \frac{3\pm\sqrt{5}}{2}$.

As real symmetric matrices are Hermitian, we see that they also have real eigenvalues. Further, the computation above applied to an anti-Hermitian matrix would give $\lambda^* = -\lambda$, which implies that the eigenvalues are purely imaginary (i.e., their real part is zero). Summarizing, we have

**Corollary 24.2.**
- *Eigenvalues of Hermitian matrices are real.*
- *Eigenvalues of real symmetric matrices are real.*
- *Eigenvalues of anti-Hermitian matrices are purely imaginary. In particular, eigenvalues of real anti-symmetric matrices are purely imaginary.*

**Unitary matrices.** Now let $A \in \mathrm{Mat}_n$ be unitary, so that $A^\dagger A = I$. Let $\underline{v}$ be an eigenvector of $A$, $A\underline{v} = \lambda\underline{v}$. Let us compute $\underline{v}^\dagger\underline{v}$. We have

$$\underline{v}^\dagger\underline{v} = \underline{v}^\dagger I\underline{v} = \underline{v}^\dagger A^\dagger A\underline{v} = (A\underline{v})^\dagger(A\underline{v}) = (\lambda\underline{v})^\dagger(\lambda\underline{v}) = \underline{v}^\dagger\lambda^*\lambda\underline{v} = (\lambda\lambda^*)(\underline{v}^\dagger\underline{v}) = |\lambda|^2(\underline{v}^\dagger\underline{v}).$$

Therefore, $|\lambda|^2 = 1$, so $|\lambda| = 1$.

**Corollary 24.3.** *Eigenvalues of unitary and real orthogonal matrices have modulus* 1.

**Example 24.4.** Let $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Then $\lambda = \pm e^{i\varphi}$, so $|\lambda| = 1$.

## Diagonalization

A matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ is said to be *diagonalizable* if we can choose a basis for $\mathbb{C}^n$ consisting of eigenvectors of $L_A$.

The term *diagonalization* means the following. Given a basis $\{\underline{v}_1, \ldots, \underline{v}_n\}$ for $\mathbb{C}^n$, we can create a *matrix* $B = (b_{ij})$ of $L_A$ *with respect to this basis*: we can write for every $j = 1, \ldots, n$

$$L_A(\underline{v}_j) = b_{1j}\underline{v}_1 + b_{2j}\underline{v}_2 + \cdots + b_{nj}\underline{v}_n = \sum_{i=1}^{n} b_{ij}\underline{v}_i.$$

If the basis consists of eigenvectors of $L_A$, then $L_A\underline{v}_j = \lambda_j\underline{v}_j$, so the matrix $B$ is *diagonal*.

**Example 24.5.** We have already seen in (22.4) that $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ has eigenvectors

$$\underline{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \ \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Therefore,

$$L_A\underline{v}_1 = 1 \cdot \underline{v}_1 + 0 \cdot \underline{v}_2, \quad L_A\underline{v}_2 = 0 \cdot \underline{v}_1 + 2 \cdot \underline{v}_2,$$

and thus $L_A$ is given by the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is the basis $\{\underline{v}_1, \underline{v}_2\}$.

## Lecture 25

**Example 25.1** (Diagonalization of $2 \times 2$ matrices)**.** If we have a $2 \times 2$ matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and choose a basis $\underline{v}_1, \underline{v}_2$ for $\mathbb{C}^2$, then the matrix for $L_A$ w.r.t. this basis is given as follows. We can write

$$\begin{aligned} L_A(\underline{v}_1) &= A\underline{v}_1 = \alpha_1 \cdot \underline{v}_1 + \alpha_2 \cdot \underline{v}_2, \\ L_A(\underline{v}_2) &= A\underline{v}_2 = \alpha_3 \cdot \underline{v}_1 + \alpha_4 \cdot \underline{v}_2 \end{aligned} \tag{*}$$

for some uniquely determined $\alpha_i \in \mathbb{C}$ (because $\underline{v}_1, \underline{v}_2$ is a basis, so any vector is a unique linear combination of $\underline{v}_1, \underline{v}_2$). Now, the matrix for $L_A$ w.r.t. this basis is

$$\begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{pmatrix}.$$

So, if $A$ is diagonalizable, it means that we can choose $\underline{v}_1, \underline{v}_2$ such that $\alpha_2 = \alpha_3 = 0$ (which means exactly that $\underline{v}_1$ ($\underline{v}_2$, resp.) is an eigenvector for $A$ with eigenvalue $\alpha_1$ ($\alpha_4$, resp.).

Now, if $A$ is diagonalizable, we can form a matrix $P$ whose columns are the basis vectors $\underline{v}_1, \underline{v}_2$ (who are the eigenvectors of $A$ according to our choice). That is, if we write

$$\underline{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} z \\ w \end{pmatrix},$$

for some coordinates $x, y, z, w$, then

$$P = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$$

and equations $(*)$ become

$$\begin{aligned} A\begin{pmatrix} x \\ y \end{pmatrix} = \alpha_1 \begin{pmatrix} x \\ y \end{pmatrix} &\iff \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \alpha_1 \begin{pmatrix} x \\ y \end{pmatrix} \\ A\begin{pmatrix} z \\ w \end{pmatrix} = \alpha_4 \begin{pmatrix} z \\ w \end{pmatrix} &\iff \begin{pmatrix} az+bw \\ cz+dw \end{pmatrix} = \alpha_4 \begin{pmatrix} z \\ w \end{pmatrix}, \end{aligned}$$

so

$$AP = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} ax+by & az+bw \\ cx+dy & cz+dw \end{pmatrix} = \begin{pmatrix} \alpha_1 x & \alpha_4 z \\ \alpha_1 y & \alpha_4 w \end{pmatrix} = \begin{pmatrix} x & z \\ y & w \end{pmatrix}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_4 \end{pmatrix} = P\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_4 \end{pmatrix}.$$

In fact, one can show that $P$ has an inverse: indeed, the columns of $P$ are linearly independent (since $\underline{v}_1, \underline{v}_2$ form a basis of $\mathbb{C}^2$), and thus the rank of $P$ is equal to 2, which implies that $\det P \neq 0$ and thus $P$ is invertible (see Corollary 21.6). Therefore, we can multiply by $P^{-1}$ on both sides to obtain

$$P^{-1}AP = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_4 \end{pmatrix}.$$

Thus, we come to the following **algorithm of diagonalization** of $2 \times 2$ matrices.

**Step 1**: Compute charasteritic polynnomial of $A$ and eigenvalues $\lambda_1$ and $\lambda_2$.

**Step 2**: Find two linearly independent eigenvectors $\underline{v}_1$ and $\underline{v}_2$ of $A$.

**Step 3**: Compose a matrix $P$ (which is called a *transformation matrix*) whose columns are $\underline{v}_1$ and $\underline{v}_2$. Then $P^{-1}AP$ is diagonal with diagonal entries $\lambda_1$ and $\lambda_2$.

*Remark.*

- In Step 1, the eigenvalues $\lambda_1$ and $\lambda_2$ may coincide.
- In Step 2, we may not be able to find two linearly independent eigenvectors $\underline{v}_1$ and $\underline{v}_2$ of $A$. Then the whole procedure fails, which means that the matrix is not diagonalizable.
- If $\lambda_1 \neq \lambda_2$ then eigenvectors $\underline{v}_1$ and $\underline{v}_2$ are linearly independent, so the matrix is diagonalizable. Indeed, assuming that $\underline{v}_1$ and $\underline{v}_2$ are linearly dependent we conclude that they belong to the same eigenspace, and thus have the same eigenvalue, which leads to a contradiction.

**Example 25.2.** $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Then the characteristic polynomial is $\det(A - \lambda I) = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = (\lambda - 2)\lambda$, so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. Solving the homogenuous systems $(A - \lambda_i I)\underline{v}_i = \underline{0}$, we find the corresponding eigenspaces, and then we choose one eigenvector from each: we can take, for example, $\underline{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Then we get a matrix $P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ whose columns are $\underline{v}_1$ and $\underline{v}_2$, and one can check that

$$P^{-1}AP = \left( \frac{1}{-2i} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

All of this works similarly for any $n \times n$ matrix for $n \geq 3$.

We summarize the argument from Example 25.1:

**Proposition 25.3.** *Let $A \in \mathrm{Mat}_n(\mathbb{C})$ be diagonalizable, i.e. there is a basis $\underline{v}_1, \ldots, \underline{v}_n$ for $\mathbb{C}^n$ such that each $\underline{v}_i$ is an eigenvector for $A$. If $P$ is the matrix whose columns are the vectors $\underline{v}_1, \ldots, \underline{v}_n$, then*

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

*where $\lambda_i$ is the eigenvalue for the eigenvector $\underline{v}_i$.*

The essential requirement in the above proposition is that the vectors $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent (they will then be a basis). If $A$ has $n$ *distinct* eigenvalues (as we had in (22.4) and (23.5)), then it will have $n$ linearly independent eigenvectors, and hence be diagonalizable. This is because of the following fact:

**Fact.** Eigenvectors with distinct eigenvalues are linearly independent.

**Corollary 25.4.** *If $A \in \mathrm{Mat}_n(\mathbb{C})$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.*

Indeed, by the definition of an eigenvalue, for every eigenvalue $\lambda_i$ we can find a non-zero eigenvector $\underline{v}_i$. Due to the fact above, they all are linearly independent, and since there are $n$ of them they compose a basis of $\mathbb{C}^n$.

*Remark.* According to the Fundamental Theorem of Algebra, any complex polynomial in one variable of degree $n$ always has $n$ roots (some of which may coincide). In particular, this can be applied to the characteristic polynomial of $A \in \mathrm{Mat}_n(\mathbb{C})$. In other words, if $\det(A - \lambda I)$ has $k$ distinct roots, we can write

$$\det(A - \lambda I) = (\lambda_1 - \lambda)^{l_1}(\lambda_2 - \lambda)^{l_2} \dots (\lambda_k - \lambda)^{l_k} = \Pi_{i=1}^{k}(\lambda_i - \lambda)^{l_i},$$

where the sum of all $l_i$ is equal to $n$ (numbers $l_i$ are called *multiplicities* of roots $\lambda_i$).

# Lecture 26

We have seen that we can easily diagonalize a matrix having $n$ distinct eigenvalues. Even if the eigenvalues are not distinct, a matrix may still be diagonalizable:

**Example 26.1.** Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & -2 & 0 \end{pmatrix}$. Its eigenvalues are given by the roots of the polynomial

$$(1-\lambda)((-1-\lambda)(-\lambda)-2) = (1-\lambda)(\lambda^2+\lambda-2) = (1-\lambda)^2(\lambda+2),$$

that is, eigenvalues: $1, -2$.
We now find the eigenspaces. For $\lambda = 1$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & -1 \\ 1 & -2 & -1 \end{pmatrix} \underline{x} = 0 \Longleftrightarrow x - 2y - z = 0.$$

Thus, the eigenspace has two free parameters:

$$E_1 = \left\{ \begin{pmatrix} 2y+z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{C} \right\}.$$

Thus $E_1$ is a two-dimensional space, so we can find two linearly independent vectors in it, for example

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$E_1 = \left\{ y\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{C} \right\} = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Moreover, for $\lambda = -2$, we get

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -2 & 2 \end{pmatrix} \underline{x} = 0 \Longleftrightarrow \begin{pmatrix} 0 & 6 & -6 \\ 0 & 3 & -3 \\ 1 & -2 & 2 \end{pmatrix} \underline{x} = 0$$

$$\Longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = 0 \Longleftrightarrow \begin{cases} x & = 0 \\ y - z & = 0. \end{cases} \Longleftrightarrow \underline{x} = \begin{pmatrix} 0 \\ y \\ y \end{pmatrix}, \; y \in \mathbb{C}.$$

Thus $E_{-2} = \left\{ \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \mid y \in \mathbb{C} \right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$

Since $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for an eigenvalue which is distinct from that for the eigenvectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

and $\begin{pmatrix}1\\0\\1\end{pmatrix}$, we see that

$$\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$$

are linearly independent. Thus, they form a basis for $\mathbb{C}^3$ and hence $A$ is diagonalizable.

In fact, if we form the matrix

$$P = \begin{pmatrix}2 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix},$$

then Proposition 25.3 tells us that

$$P^{-1}AP = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -2\end{pmatrix}.$$

Summarizing, we just need to choose a basis in each eigenspace, and then collect all of these together to compose a basis of the whole space $\mathbb{C}^n$. This can be done in the only case when we have "enough" linearly independent eigenvectors in every eigenspace, i.e. if the sum of all of the dimensions $\dim E_{\lambda_i}$ is equal to $n$. In fact, the following (non-trivial) statement always holds.

**Fact.** Let $\det(A - \lambda I) = \Pi_{i=1}^{k}(\lambda_i - \lambda)^{l_i}$, where the sum of all $l_i$ is equal to $n$ (recall that numbers $l_i$ are called *multiplicities* of roots $\lambda_i$). Then for every $i = 1, \ldots, k$ one has $\dim E_{\lambda_i} \leq l_i$.

Therefore, the example above can be generalized in the following way:

**Theorem 26.2.** *Let $A \in \mathrm{Mat}_n(\mathbb{C})$, let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $A$, and let the characteristic polynomial of $A$ be $\det(A - \lambda I) = \Pi_{i=1}^{k}(\lambda_i - \lambda)^{l_i}$, where the sum of all $l_i$ is equal to $n$. Then $A$ is diagonalizable if and only if for every $i = 1, \ldots, k$ the dimension of the eigenspace $E_{\lambda_i}$ is equal to $l_i$.*

Note that, by definition, $E_{\lambda_i} = \ker(A - \lambda_i I)$, and thus $\dim E_{\lambda_i} = n - \mathrm{rk}(A - \lambda_i I)$. Thus, Theorem 26.2 can be reformulated in the following easy-to-use way:

**Corollary 26.3.** *A matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ with eigenvalues $\lambda_1, \ldots, \lambda_k$ and characteristic polynomial $\det(A - \lambda I) = \Pi_{i=1}^{k}(\lambda_i - \lambda)^{l_i}$ is diagonalizable if and only if for every $i = 1, \ldots, k$ we have $n - \mathrm{rk}(A - \lambda_i I) = l_i$.*

Corollary 26.3 leads to the following algorithm.

**Criterion of diagonalizability of a matrix.** Let $A \in \mathrm{Mat}_n(\mathbb{C})$. To decide whether $A$ is diagonalizable, we need to do the following.

**Step 1**. Compute the characteristic polynomial $\det(A - \lambda I) = (\lambda_1 - \lambda)^{l_1}(\lambda_2 - \lambda)^{l_2} \ldots (\lambda_k - \lambda)^{l_k}$.

**Step 2**. For every $i = 1, \ldots, k$ compute the number $n - \mathrm{rk}(A - \lambda_i I)$.

**Step 3**. If for *every* $i = 1, \ldots, k$ we have $n - \mathrm{rk}(A - \lambda_i I) = l_i$, then $A$ is diagonalizable. Otherwise, it is not.

*Remark.* In Steps 2 and 3, we need to consider only eigenvalues $\lambda_i$ with $l_i > 1$. Indeed, since we know that $1 \leq n - \mathrm{rk}(A - \lambda_i I) \leq l_i$, the equality $l_i = 1$ guarantees that $n - \mathrm{rk}(A - \lambda_i I) = 1 = l_i$ (in particular, we immediately get Corollary 25.4).

# Lecture 27

### Computing high powers of a diagonalizable matrix

A typical problem about matrices is: Given a matrix, for example, $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$, compute

$$A^{1000} = \underbrace{A \cdot A \cdots A}_{1000 \text{ times}}.$$

If we start by simply trying to multiply $A$ with itself:

$$A^2 = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ -6 & -2 \end{pmatrix}$$
$$A^3 = \cdots$$

we quickly realize that this is going to be a lot of work, even for a computer. However, if $A$ is diagonalizable, which we know that this $A$ is (by (22.4)), it means that there is a matrix $P$ such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix},$$

and it is easy to take a large power of a diagonal matrix:

$$(P^{-1}AP)^{1000} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}^{1000} = \begin{pmatrix} \lambda_1^{1000} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^{1000} \end{pmatrix}.$$

Now, the left hand side is

$$(P^{-1}AP)^{1000} = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) = P^{-1}A^{1000}P$$

(note that all the "inner" $PP^{-1}$ cancel, because $PP^{-1} = I$). So

$$P^{-1}A^{1000}P = \begin{pmatrix} \lambda_1^{1000} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^{1000} \end{pmatrix},$$

and thus

$$A^{1000} = P \begin{pmatrix} \lambda_1^{1000} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^{1000} \end{pmatrix} P^{-1},$$

which we can compute, if we know $P$.

Similarly, we can compute a power series of a diagonalizable matrix $A$, e.g. exponent: if we denote the diagonal matrix $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$ as $\Lambda$, we have

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{(P\Lambda P^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{P\Lambda^k P^{-1}}{k!} = P(\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!})P^{-1} = P\exp(\Lambda)P^{-1},$$

where

$$\exp(\Lambda) = \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^k \end{pmatrix} = \begin{pmatrix} e_1^\lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_n^\lambda \end{pmatrix}.$$

**Example 27.1.** Let's take $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$, as above. By (22.4) we know that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

with $P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$. Thus, by the above discussion,

$$A^{1000} = P \begin{pmatrix} 1 & 0 \\ 0 & 2^{1000} \end{pmatrix} P^{-1}.$$

So to compute this, we need $P^{-1}$:

$$\left( \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{array} \right) \xrightarrow{R_2+2R_1} \left( \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array} \right) \xrightarrow{R_1-R_2} \left( \begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \end{array} \right),$$

so

$$P^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} A^{1000} &= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{1000} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^{1000} \\ -2 & -2^{1000} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{1001} - 1 & 2^{1000} - 1 \\ 2 - 2^{1001} & 2 - 2^{1000} \end{pmatrix}. \end{aligned}$$

The entries here are quite large numbers, whose decimal expansions could be calculated, but are best just left as they are.

Now,

$$\exp(A) = P \exp \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} e^1 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2e^2 - e & e^2 - e \\ 2e - e^2 & 2e + e^2 \end{pmatrix}.$$

Let's take a $3 \times 3$ as well:

**Example 27.2.** Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & -2 & 0 \end{pmatrix}$. We will compute the $n$th power of $A$, for an arbitrary integer $n \geq 1$. By (26.1), we know that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

where

$$P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We need $P^{-1}$. Calculating it via Gauss elimination as before, gives

$$P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

Thus, for each $n$,

$$\begin{aligned} A^n &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-2)^n \end{pmatrix} \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & (-2)^n \\ 0 & 1 & (-2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 1-(-2)^n & 1+2(-2)^n & -1+2(-2)^n \\ 1-(-2)^n & -2+2(-2)^n & 2+(-2)^n \end{pmatrix}. \end{aligned}$$

Note that $(-2)^n = (-1)^n 2^n = \begin{cases} 2^n & \text{if } n \text{ is even} \\ -2^n & \text{if } n \text{ is odd} \end{cases}.$

Remark: It is only for diagonalizable matrices we can do this. For non-diagonalizable matrices the powers can also be calculated by using *Jordan normal form.*