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Epiphany 2019

**Geometry III/IV, Hints: weeks 11–12**

## Möbius transformations, inversion

**11.1.** One way is a direct computation. Another possibility is to use the correspondence between $f(z) = \frac{az+b}{cz+d}$ and a multiplication by matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

**11.3.** (a) Is almost evident, but still try to find some argument.

(b) Map some intersection points to where they need to be, then use (a).

**11.4.** Draw two lines through $P$ intersecting the circle, and find similar triangles.

**11.5.** Similar to 11.4.

**12.1.** Divide by $AC \cdot BC$, find (almost) two cross-ratios on the left. Then show that in case of the inscribed quadrilateral these things are exactly the cross-ratios. Finally, use results of Problem 7.8 to get the statement.

**12.2.** (b) Use the altitude in a right-angled triangle (E23).

(c) Use the result of Question 11.4 to find the distance $PQ$ (where $Q \in \gamma$ is the point of contact).

(d) Use the tangent line constructed in (c) and consider similar right triangles.

(e) Invert the construction in (d).

(f) Use (b).

(g) If the circles are of different sizes, then there is a homothety with positive coefficient mapping one circle to another (why?). Find the centre of the homothety and use (c).

(h) Use (g) to find the centre and (b) to find the radius.

(i) Two equal circles may be swapped by reflection $r$ in a line. Two different ones may be swapped by an inversion $I_0$ found in (h). If $I$ is an inversion which takes two different circles to two equal ones, then $r = I \circ I_0 \circ I$. Use this for guessing the candidate for $I$ and for proving that $I(\gamma_1)$ and $I(\gamma_2)$ are of the same size.

**12.3.** By definition (look at the fixpoints).

**12.4.** Each of these transformations is a composition of two reflections/inversions.

**12.5.** Cf. the proof of Theorem 5.12.

**12.6.** Use cross-ratio.

**12.7.** Go through all other types: they will be easily excluded by different reasons.

**12.8.** Take a parabolic transformation preserving $\infty$ and conjugate it by a transformation mapping 1 to $\infty$.

**12.9.** Map three boundary points of the disc to three boundary points of the new domain (and don't forget to check that you obtained the correct half-plane).

**12.10.** Look at the image of $\infty$ under the iterated applications of the composition.