Durham University
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Epiphany 2019

Geometry III/IV, Homework: weeks 11–12

Due date for starred problems: **Friday, February 8**.

Möbius transformations, inversion

11.1. Show that Möbius transformations form a group.

11.2. Find a Möbius transformation which takes $1, 2, 3$ to $0, 1, \infty$.

11.3. ($\star$)

(a) Let l be a line and γ be a circle. Show that γ is orthogonal to l if and only if l contains the centre of γ.

(b) Let γ_1, γ_2, γ_3 be three mutually orthogonal circles on the plane. Show that there exists a Möbius transformation which takes them to the curves $\{x = 0\}$, $\{y = 0\}$ and $x^2 + y^2 = 1$.

11.4. ($\star$) Let γ be a circle and P be a point lying outside of γ. Let l be a line through P intersecting γ, denote by A and B the intersection points of l with γ. Show that the product $|PA| \cdot |PB|$ does not depend on the choice of l. (This product is also called the *power of P with respect to γ*).

11.5. The same question as 11.4, but P lies inside γ.

12.1. Prove the theorem of Ptolemy: for a cyclic quadrilateral $ABCD$, the following equality holds:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

12.2. ($\star$) (Inversion with ruler and compass).
In this question, you need to find and describe algorithms for certain constructions as well as to justify them. You can use without proofs and further descriptions the following constructions:

- midpoint of a given segment;
- perpendicular bisector for a given segment;
- the line perpendicular to a given line through a given point.

This is a long question with an easy start and more complicated parts at the end. Please, try to submit parts **(b)–(e)**. You are welcome to submit solutions/sketches for later parts, but formally it is not the part of the written assignment.

(a) Given a circle γ, construct its centre.

(b) Given segments of length a and b construct a segment of length h satisfying $h^2 = a \cdot b$.

(c) Given a circle γ and a point P outside the circle, construct a line PQ tangent to γ.

(d) Given a circle γ and a point A outside the circle, construct the inversion image of A

(e) Construct the inversion image for the point A' lying inside the circle γ.

(f) Let O, A' and A be three points lying on a line (A' lies between O and A).
Construct a circle γ centred at O such that the inversion with respect to γ takes A to A'.

(g) Given two circles γ_1 and γ_2, construct a line tangent to both of them.

(h) Given two circles γ_1 and γ_2 of different sizes, construct an inversion which takes γ_1 to γ_2 and takes γ_2 to γ_1.
(You need to construct the centre and the radius of the circle of inversion).

(i) Given two circles γ_1 and γ_2 of different sizes,
find an inversion which takes them to a pair of equal circles.
(You need to construct the centre and the radius of the circle of inversion).

12.3. What type is the transformation $1/z$?
(Hint: parabolic or not? if not, then is it elliptic, or hyperbolic, or loxodromic?)

12.4. Write the following transformations as compositions of reflections/inversions:
(a) $2z$ (b) $-z$ (c) $z+1$ (d) $\frac{1}{z}$

12.5. Let I be an inversion with respect to the unit circle $|z| = 1$. Find the image $I(l)$ of the line l given by the equation $\operatorname{Re} z = 2$.

12.6. Do the points $-1-2i$, $-1+2i$, $3+i$, $3-i$ lie on one line or circle?

12.7. Show that a finite order Möbius transformation is elliptic.
(g is called *of finite order* if $g^n = id$ for some integer n).

12.8. Find a parabolic Möbius transformation preserving the point $z = 1$.

12.9. Find a Möbius transformation mapping the disc $|z| < 1$ to the half-plane $\operatorname{Re} z > 2$.

12.10. Let I_0 be the inversion with respect to the circle $|z| = 1$, and I_1 be the inversion with respect to the circle $|z-1| = 1$. What is the type of the Möbius transformation obtained as a composition $I_1 \circ I_0$?
(Hint: try to find a geometric solution without writing any formulae).

References:

1. Lectures (Möbius transformations, Inversion, Stereographic projection) and Lecture V in Prasolov's book.
2. Section 1.1 of the book *Hyperbolic Geometry* by Caroline Series.
3. For introduction and discussion of inversion you can check the following sources (both linked from the course on-line resources page):
   - *Circle inversion* by Malin Christersson.
   - *Inversion in a circle* by Tom Davis.