Durham University
Pavel Tumarkin

Epiphany 2019

Geometry III/IV, Homework: weeks 13–14

Due date for starred problems: **Friday, February 22**.

Hyperbolic geometry: conformal models

13.1. Draw in each of the two conformal models (Poincaré disc and upper half-plane):

(a) two intersecting lines;
(b) two parallel lines;
(c) two ultra-parallel lines;
(d) infinitely many disjoint half-planes;
(e) a circle tangent to a line.

13.2. In the upper half-plane model draw

(a) a line through the points i and $i+2$;
(b) a line through $i+1$ orthogonal to the line represented by the ray $\{ki \mid k>0\}$;
(c) a circle centred at i (just sketch it, no formula needed);
(d) a triangle with all three vertices at the absolute (such a triangle is called *ideal*).

13.3. Prove SSS, ASA and SAS theorems of congruence of hyperbolic triangles.

13.4. Let ABC be a triangle. Let $B_1 \in AB$ and $C_1 \in AC$ be two points such that $\angle AB_1C_1 = \angle ABC$. Show that $\angle AC_1B_1 > \angle ACB$.

13.5. Show that there is no "rectangle" in hyperbolic geometry (i.e. no quadrilateral has four right angles).

13.6. ($\star$) Given an acute-angled polygon P (i.e. a polygon with all angles smaller or equal to $\pi/2$) and lines m and l containing two disjoint sides of P, show that l and m are ultra-parallel.

14.1. Given non-negative real numbers α, β, γ such that $\alpha+\beta+\gamma<\pi$, show that there exists a hyperbolic triangle with angles α, β, γ.

14.2. Show that there exists a hyperbolic pentagon with five right angles.

14.3. An *ideal* triangle is a hyperbolic triangle with all three vertices on the absolute.

(a) Show that all ideal triangles are congruent.
(b) Show that the altitudes of an ideal triangle are concurrent.
(c) Show that an ideal triangle has an inscribed circle.

14.4. It was proved in lectures that an isometry fixing 3 points of the absolute is the identity map. How many isometries fix two points of the absolute? Classify the isometries fixing 0 and ∞ in the upper half-plane model.

14.5. ($\star$)

(a) Show that the group of isometries of the hyperbolic plane is generated by reflections.

(b) How many reflections do you need to map a triangle ABC to a congruent triangle $A'B'C'$?

14.6. ($\star$)

(a) Does there exist a regular triangle on hyperbolic plane?

(b) Does there exist a right-angled regular polygon on hyperbolic plane? How many edges does it have (if exists)?

14.7. (a) Show that the angle bisectors in a hyperbolic triangle are concurrent.

(b) Show that every hyperbolic triangle has an inscribed circle.

(c) Does every hyperbolic triangle have a circumscribed circle?

References:

Lectures (Conformal models of hyperbolic plane; Elementary hyperbolic geometry) are based on Lectures VI and VII in Prasolov's book.