Durham University
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Epiphany 2019

Geometry III/IV, Homework: weeks 15–16

Due date for starred problems: **Friday, March 8**.

Elementary hyperbolic geometry

15.1. (a) Let P and Q be the feet of the altitudes in an ideal hyperbolic triangle. Find PQ.

(b) Find the radius of a circle inscribed into an ideal hyperbolic triangle.

(c) Show that a radius of a circle inscribed into a hyperbolic triangle does not exceed $\operatorname{arccosh}(2/\sqrt{3})$.

15.2. For a right hyperbolic triangle ($\gamma = \frac{\pi}{2}$) show:

$$\text{(a)}\, \tanh b = \tanh c \cos\alpha, \qquad \text{(b)}\, \sinh a = \sinh c \sin\alpha.$$

15.3. Show that in the upper half-plane model the following distance formula holds:

$$4\sinh^2\frac{d}{2} = \frac{|z-w|^2}{\mathrm{Im}(z)\mathrm{Im}(w)}.$$

15.4. Find an area of a right-angled hyperbolic pentagon.

15.5. ($\star$) In the upper half-plane model, find the locus of points that lie on distance d from the line $\{\mathrm{Re}\, z = 0\}$.

Projective models

16.1. In the Klein disc model draw two parallel lines, two ultra-parallel lines, an ideal triangle, a triangle with angles $(0, \frac{\pi}{2}, \frac{\pi}{3})$.

16.2. ($\star$) Show that three altitudes of a hyperbolic triangle either have a common point, or are all parallel to each other, or there exists a unique line orthogonal to all three altitudes.

16.3. Let $\boldsymbol{u}, \boldsymbol{v}$ be two vectors in $\mathbb{R}^{2,1}$. Denote $Q = |\frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}|$, where $(x, y) = x_1y_1 + x_2y_2 - x_3y_3$. Show the following distance formulae:

(a) if $(\boldsymbol{u},\boldsymbol{u}) < 0$, $(\boldsymbol{v},\boldsymbol{v}) < 0$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ define two points in $\mathbb{H}^2$, and $Q = \cosh^2 d(\boldsymbol{u},\boldsymbol{v})$.

(b) if $(\boldsymbol{u},\boldsymbol{u}) < 0$, $(\boldsymbol{v},\boldsymbol{v}) > 0$, then $\boldsymbol{u}$ defines a point and $\boldsymbol{v}$ defines a line $l_{\boldsymbol{v}}$ in $\mathbb{H}^2$, and $Q = \sinh^2 d(\boldsymbol{u}, l_{\boldsymbol{v}})$.

(c) if $(\boldsymbol{u},\boldsymbol{u}) > 0$, $(\boldsymbol{v},\boldsymbol{v}) > 0$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ define two lines $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ in $\mathbb{H}^2$ and

- if $Q < 1$, then $l_{\boldsymbol{u}}$ intersects $l_{\boldsymbol{v}}$ forming angle φ satisfying $Q = \cos^2\varphi$;
- if $Q = 1$, then $l_{\boldsymbol{u}}$ is parallel to $l_{\boldsymbol{v}}$;
- if $Q > 1$, then $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ are ultra-parallel lines satisfying $Q = \cosh^2 d(l_{\boldsymbol{u}}, l_{\boldsymbol{v}})$.

16.4. ($\star$) Consider the two-sheet hyperboloid model $\{\boldsymbol{u} = (u_1, u_2, u_3) \in \mathbb{R}^{2,1} \mid (\boldsymbol{u}, \boldsymbol{u}) = -1,\ u_3 > 0\}$, where $(\boldsymbol{u}, \boldsymbol{u}) = u_1^2 + u_2^2 - u_3^2$.

(a) For the vectors

$$\begin{array}{lll} \boldsymbol{v}_1 = (2,1,2) & \boldsymbol{v}_2 = (0,1,2) & \boldsymbol{v}_3 = (3,4,5) \\ \boldsymbol{v}_4 = (1,0,0) & \boldsymbol{v}_5 = (0,1,0) & \boldsymbol{v}_6 = (1,1,2) \end{array}$$

decide whether $\boldsymbol{v}_i$ defines a point in $\mathbb{H}^2$, a point on the absolute, or a line in $\mathbb{H}^2$.

(b) Find the distance between the two points of $\mathbb{H}^2$ described in (a).

(c) Which pairs of lines in (a) are intersecting? Which lines are parallel? Which lines are ultra-parallel? Justify your answer.

(d) Find the distances between all pairs of ultra-parallel lines in (a).

(e) Does any of the points in (a) lie on any of the lines above?

(f) Find the angles between the pairs of intersecting lines.

References:

Lectures (Elementary hyperbolic geometry, area, Klein model and hyperboloid model) are based on Lectures VII, VIII, VI and XIII of Prasolov's book.