Durham University
Pavel Tumarkin

Epiphany 2019

Geometry III/IV, Homework: weeks 17–18

Due date for starred problems: **Tuesday, March 19**.

Isometries of the hyperbolic plane

17.1. Show that any pair of parallel lines can be transformed to any other pair of parallel lines by an isometry.

17.2. Let $A, B \in \gamma$ be two points on a horocycle γ. Show that the perpendicular bisector to the segment AB (of a hyperbolic line!) is orthogonal to γ.

17.3. Let f be a composition of three reflections. Show that f is a glide reflection, i.e. a hyperbolic translation along some line composed with a reflection with respect to the same line.

17.4. ($\star$) Let f be an isometry of the hyperbolic plane such that the distance from A to $f(A)$ is the same for all points $A \in \mathbb{H}^2$. Show that f is an identity map.

17.5. ($\star$) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors in the hyperboloid model such that $(\boldsymbol{a}, \boldsymbol{a}) > 0$ and $(\boldsymbol{b}, \boldsymbol{b}) > 0$. Let $l_{\boldsymbol{a}}$ and $l_{\boldsymbol{b}}$ be the lines determined by equations $(\boldsymbol{x}, \boldsymbol{a}) = 0$ and $(\boldsymbol{x}, \boldsymbol{b}) = 0$ respectively, and let $r_{\boldsymbol{a}}$ and $r_{\boldsymbol{b}}$ be the reflections with respect to $l_{\boldsymbol{a}}$ and $l_{\boldsymbol{b}}$.

(a) For $\boldsymbol{a} = (0, 1, 0)$ and $\boldsymbol{b} = (1, 0, 0)$ write down $r_{\boldsymbol{a}}$ and $r_{\boldsymbol{b}}$. Find $r_{\boldsymbol{b}} \circ r_{\boldsymbol{a}}(\boldsymbol{v})$, where $\boldsymbol{v} = (0, 1, 2)$.

(b) What is the type of the isometry $\varphi = r_{\boldsymbol{b}} \circ r_{\boldsymbol{a}}$ for $\boldsymbol{a} = (1, 1, 1)$ and $\boldsymbol{b} = (1, 1, -1)$?
(Hint: you don't need to compute $r_{\boldsymbol{a}}$ and $r_{\boldsymbol{b}}$).

(c) Find an example of $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\varphi = r_{\boldsymbol{b}} \circ r_{\boldsymbol{a}}$ is a rotation by $\pi/2$.

Equidistant curves

18.1. Let l be a hyperbolic line, and let E_l be an equidistant curve for l.

(a) Let C_1 and C_2 be two connected components of the same equidistant curve E_l. Show that that C_1 is also equidistant from C_2, i.e. given a point $A \in C_1$ the distance $d(A, C_2)$ from A to C_2 does not depend on the choice of A.

(b) Let $A \in E_l$ be a point on the equidistant curve, and let $A_l \in l$ be the point of l closest to A. Show that the line AA_l is orthogonal to the equidistant curve E_l.

(c) Let $P, Q \in l$ be two points on l, and let $A \in E_l$. Continue the rays AP and AQ till the next intersection points with E_l, denote the resulting intersection points by B and C. Let T be a curvilinear triangle ABC (with geodesic sides AB and AC, but BC being a segment of the equidistant curve). Assuming that all angles of ABC are acute, show that the area of T does not depend on the choice of $A \in E_l$.

(d) In the assumptions of (c), show that the area of the geodesic triangle ABC does not depend on the choice of A.

18.2. ($\star$)

(a) Let l and l' be ultra-parallel lines. Let E_l be an equidistant curve for l intersecting l' in two points A and B. Denote by h the common perpendicular to l and l' and let $H = h \cap l'$ be the intersection point. Show that $AH = HB$.

(b) Let l be a line and E_l be an equidistant curve for l. For two points A, B on E_l, show that the perpendicular bisector of AB is also orthogonal to l.

(c) Let ABC be a triangle in the Poincaré disc model. Let γ be a Euclidean circumscribed circle (i.e. a circumscribed circle for ABC considered as a Euclidean triangle). Suppose that γ intersects the absolute at points X and Y. Show that the (hyperbolic) perpendicular bisector to AB is orthogonal to the hyperbolic line XY.

(d) Show that three perpendicular bisectors in a hyperbolic triangle are either concurrent, or parallel, of have a common perpendicular.

References:

1. Lectures on types of isometries, horocycles and equidistant curves are based on Lecture IX of Prasolov's book.

2. Lectures on Poincaré theorem and modular group are based on
• parts of "Hyperbolic geometry" Caroline Series, especially some parts of Chapter 6, as well as (small pieces of) Chapters 4 and 5;
• and on Lectures X and XI of Prasolov's book.