Durham University
Pavel Tumarkin

Epiphany 2019

Geometry III/IV, Term 2 (Section 5)

5 Möbius geometry

5.1 Group of Möbius transformations

Definition 5.1. A map $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given by $f(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ is called a *Möbius transformation* or a *linear-fractional transformation*.

Remark. It is a bijection of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto itself.

Theorem 5.2. (a) Möbius transformations form a group (denote it Möb), this group is isomorphic to $PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/\{\lambda I \mid \lambda \neq 0\}$.

(b) Möb is generated by $z \mapsto az$, $z \mapsto z + b$ and $z \mapsto 1/z$, $a, b \in \mathbb{C} \setminus \{0\}$.

Example. Matrices of the generators of Möb.

Theorem 5.3. (a) Möbius transformations act on $\overline{\mathbb{C}}$ triply-transitively.

(b) A Möbius transformation is uniquely determined by the images of any three distinct points.

Theorem 5.4. Möbius transformations

(a) take lines and circles to lines and circles and

(b) preserve angles between curves.

Example. The map $z \mapsto 1/z$ preserves the line $\operatorname{Im} z = 0$ and takes the circle $|z - 1/2| = 1/2$ to the line $\operatorname{Re} z = 1$.

5.2 Types of Möbius transformations

Definition 5.5. A Möbius transformation with a unique fixed point is called *parabolic*.

Example. A map $z \mapsto z + b$, $b \in \mathbb{C}^*$, is parabolic.

Proposition 5.6. Every parabolic Möbius transformation is conjugate in Möb to $z \mapsto z + 1$.

Proposition 5.7. Every non-parabolic Möbius transformation is conjugate in Möb to $z \mapsto az$, $a \in \mathbb{C} \setminus \{0\}$.

Definition 5.8. A non-parabolic Möbius transformation conjugate to $z \mapsto az$ is called
(1) *elliptic*, if $|a| = 1$; (2) *hyperbolic*, if $|a| \neq 1$ and $a \in \mathbb{R}$; (3) *loxodromic*, otherwise.

Remark. Two fixpoints of a hyperbolic or a loxodromic transformation have different properties: one is *attracting*, another is *repelling*. Elliptic transformations have two "similar" fixpoints (neither attracting nor repelling).

5.3 Inversion

Definition 5.9. Let $\gamma \subset \mathbb{C}$ be a circle with centre O and radius r. An *inversion* I_γ with respect to γ takes a point A to a point A' lying on the ray OA s.t. $|OA| \cdot |OA'| = r^2$.

Exercise. If γ is the unit circle centred at 0, then $I_\gamma(z) = 1/\bar{z}$.

Proposition 5.10. (a) $I_\gamma^2 = id$.

(b) Inversion in γ preserves γ pointwise ($I_\gamma(A) = A$ for all $A \in \gamma$).

Lemma 5.11. If $P' = I_\gamma(P)$ and $Q' = I_\gamma(Q)$ then $\triangle OPQ$ is similar to $\triangle OQ'P'$.

Theorem 5.12. Inversion takes circles and lines to circles and lines. More precisely,

(1) lines through O $\longleftrightarrow$ lines through O;

(2) lines not through O $\longleftrightarrow$ circles through O;

(3) circles not through O $\longleftrightarrow$ circles not through O.

Theorem 5.13. Inversion preserves angles.

Remark. Inversion can be understood as "reflection with respect to a circle".

Example. The inversion in the unit circle is conjugate to the reflection in $\operatorname{Re} z = -1/2$ by applying $z \mapsto 1/(z-1)$.

Exercise 5.14. Every inversion is conjugate in Möb to any reflection.

Theorem 5.15. Every Möbius transformation is a composition of an even number of inversions and reflections.

Remark. Inversions and reflections change orientation of the plane. Theorem 5.15 shows that Möbius transformations preserve orientation.

5.4 Möbius transformations and cross-ratios

Definition 5.16. For $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$, the number $[z_1, z_2, z_3, z_4] = \frac{z_3 - z_1}{z_3 - z_2} / \frac{z_4 - z_1}{z_4 - z_2} \in \overline{\mathbb{C}}$ is called the *cross-ratio.*

Remark. If the points z_1, z_2, z_3, z_4 are distinct, then $[z, z_2, z_3, z_4] \in \mathbb{C} \setminus \{0\}$.

Theorem 5.17. Möbius transformations preserve cross-ratios.

Remark. Reflections and inversions take the cross-ratio to its complex conjugate.

Proposition 5.18. Points $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$ lie on one line or circle iff $[z_1, z_2, z_3, z_4] \in \mathbb{R}$.

Proposition 5.19. Given four distinct points $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$, one has $[z_1, z_2, z_3, z_4] \neq 1$.

Corollary 5.20. For points z_1, z_2, z_3, z_4 lying on a circle or on a line, inversions and reflections preserve the cross-ratio.

5.5 Inversion in space

Definition. Let $S \subset \mathbb{R}^3$ be a sphere with centre O and radius r. An *inversion* I_S with respect to S takes a point $A \in \mathbb{R}^3 \cup \{\infty\}$ to a point A' lying on the ray OA s.t. $|OA| \cdot |OA'| = r^2$.

Theorem 5.21 (Properties of inversion)**.** Inversion

(1) takes spheres and planes to spheres and planes;

(2) takes lines and circles to lines and circles;

(3) preserves angles between curves;

(4) preserves cross-ratio of four points $[A, B, C, D] = \frac{|CA|}{|CB|} / \frac{|DA|}{|DB|}$.

5.6 Stereographic projection

Definition. Let S be a sphere centred at O, let Π be a plane through O. Let $N \in S$ be a point with $NO \perp \Pi$. The map $\pi : S \setminus N \to \Pi$ s.t. $\pi(A) = \Pi \cap NA$ for all $A \in S$ is called a *stereographic projection.*

Proposition 5.22 (Properties of stereographic projection)**.** Stereographic projection

(1) takes circles to circles and lines;

(2) preserves angles;

(3) preserves cross-ratios.

Remark. Another way to define stereographic projection is to project from N to the plane tangent to S at the point opposite to N. This projection has the same properties.