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Geometry III/IV, Term 2 (Section 6)

6 Hyperbolic geometry: conformal models

6.1 Poincaré disc model

Model: $\mathbb{H}^2$ = unit disc $D = \{|z| < 1, z \in \mathbb{C}\}$;
$\partial\mathbb{H}^2 = \{|z| = 1\}$ – boundary (called *absolute*); $\overline{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial\mathbb{H}^2$;
lines: parts of circles or lines orthogonal to $\partial\mathbb{H}^2$;
isometries: Möbius transformations, inversions, reflections (preserving the disc);
distance: a function of cross-ratio;
angles: same as Euclidean angles.

Proposition 6.1. For any two points $A, B \in \mathbb{H}^2$ there exists a unique hyperbolic line through A, B.

Remark. The same holds for $A, B \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$.

Definition 6.2. $d(A, B) = \big|\ln |[A, B, X, Y]|\big| = \left|\ln \frac{|XA|}{|XB|} / \frac{|YA|}{|YB|}\right|$, where X, Y are the points of the absolute contained in the (hyperbolic) line AB.

Theorem 6.3. $d(A, B)$ satisfies axioms of the distance.

Remark 6.4. $[A, B, X, Y]$ in Definition 6.2 is also the "usual" complex cross-ratio if we consider A, B, C, D as points in $\mathbb{C}$.

Therefore, Möbius transformations (as well as inversions and reflections) preserving the disc are indeed isometries of $\mathbb{H}^2$.

Example. Rotation about the centre of the model, reflection with respect to a diameter, inversion with respect to a circle representing a hyperbolic line are isometries of the Poincaré disc model.

Corollary 6.5. The isometry group of $\mathbb{H}^2$ acts transitively on

(a) triples of points of the absolute;

(b) points of $\mathbb{H}^2$.

Proposition 6.6. Let $l \subset \mathbb{H}^2$ be a (hyperbolic) line, $A \in \overline{\mathbb{H}}^2$ a point, $A \notin l$. Then there exists a unique line l' through A orthogonal to l.

Proposition 6.7. Let $l \subset \mathbb{H}^2$ be a (hyperbolic) line, $A \in \overline{\mathbb{H}}^2$ a point, $A \in l$. Then there exists a unique line l' through A orthogonal to l.

Proposition 6.8. For $B \in AC$, $d(A, B) + d(B, C) = d(A, C)$.

Lemma 6.9. In a right-angled triangle ABC with $\angle A = \pi/2$, $d(A, B) < d(C, B)$.

Corollary 6.10. Triangle inequality: for $B \notin AC$, $\quad d(A, B) + d(B, C) > d(A, C)$.

Remark. Triangle inequality implies that

(a) distance is well-defined, and

(b) hyperbolic lines are geodesics in the model.

Isometries of $\mathbb{H}^2$

Lemma 6.11. Hyperbolic circles are represented by Euclidean circles in the Poincaré disc model.

Remark. Isometries of $\mathbb{H}^2$ act transitively on flags.

Lemma 6.12. An isometry of $\mathbb{H}^2$ is uniquely determined by the image of any flag.

Theorem 6.13. Every isometry of the Poincaré disc model can be written as either $z \mapsto \frac{az+b}{cz+d}$ (Möbius transformation) or $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ (anti-Möbius transformation), $ad - bc \neq 0$.

Corollary. An isometry of $\mathbb{H}^2$ is uniquely determined by the images of any three points of the absolute.

Corollary. Isometries preserve angles.

Proposition 6.14. The sum of angles in a hyperbolic triangle is less than π.

Remark. If $\alpha + \beta + \gamma < \pi$ then there exists a triangle with angles α, β, γ.

Example. Let $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$. Then

$$f : z \to \frac{az+b}{\bar{b}z+\bar{a}} \in Isom(\mathbb{H}^2).$$

Remark. All Möbius isometries of the Poincaré disc model are of the form

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$$

for some $a, b \in \mathbb{C}$ with $|a|^2 - |b|^2 = 1$.

6.2 Upper half-plane model

Model: $\mathbb{H}^2 = \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$;
$\partial\mathbb{H}^2 = \{\operatorname{Im} z = 0\}$ – *absolute*;
lines: rays and half-circles orthogonal to $\partial\mathbb{H}^2$;
distance: $d(A, B) = \big|\ln |[A, B, X, Y]|\big|$;
isometries: Möbius transformations, inversions, reflections (preserving the half-plane);
angles: same as Euclidean angles.

Proposition 6.15. This defines the same geometry as Poincaré disc model.

Proposition 6.16. In the upper half-plane, hyperbolic circles are represented by Euclidean circles.

Theorem 6.17. Every isometry of the upper half-plane model can be written as either $z \mapsto \frac{az+b}{cz+d}$ or $z \mapsto \frac{a(-\bar{z})+b}{c(-\bar{z})+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$.

Remark.

- Equivalently, orientation-preserving isometries can be written as $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$; orientation-reversing isometries can be written as $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = -1$.

- Hence, for the group $Isom^+(\mathbb{H}^2)$ of orientation-preserving isometries we have $Isom^+(\mathbb{H}^2) = PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm I$.

Theorem 6.18. In the upper half-plane model, $\cosh d(z,w)) = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}$.

Remark. Lines with a common point at the absolute are called *parallel*, lines that do not intersect in $\overline{\mathbb{H}}^2$ are called *ultra-parallel* (or *divergent*).

Example. Let l, l' be parallel lines. Then $d(l,l') = 0$. (where by distance between the sets α and β we mean $d(\alpha,\beta) = \inf\limits_{A\in\alpha, B\in\beta} d(A,B)$).

6.3 Elementary hyperbolic geometry

Remark.
- In hyperbolic geometry, all Euclid's Axioms (except for the Parallel Axiom) hold.
- The Parallel Axiom for hyperbolic geometry says that there is more than one line disjoint from a given line l through a given point $A \notin l$.

Definition 6.19. For a line l and a point $A \notin l$, the *angle of parallelism* $\varphi = \varphi(A,l)$ is the half-angle between the rays emanating from A and parallel to l.

Equivalently: drop a perpendicular AH to l, then $\varphi = \angle HAQ$, $Q \in l \cap \partial\mathbb{H}^2$.

Equivalently: a ray AX from A intersects l iff $\angle HAX < \varphi$.

Remark. $\varphi(A,l)$ depends on the distance from A to l only.

Proposition 6.20. For a line l and a point $A \notin l$, let $a = d(A,l)$ and φ be the angle of parallelism. Then $\cosh a = \frac{1}{\sin\varphi}$.

Theorem 6.21 (Hyperbolic Pythagoras' theorem)**.** In a triangle with $\gamma = \pi/2$, $\quad \cosh c = \cosh a \cosh b$.

Lemma 6.22. In a triangle with a right angle γ we have $\sinh a = \sinh c \sinh \alpha$.

Theorem 6.23 (Law of sines)**.** $\quad \frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}$.

Theorem 6.24 (Law of cosines)**.** $\quad \cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha$.

Remark. For small values of a, b, c we get Euclidean sine and cosine laws.

Theorem 6.25 (Second Law of cosines)**.** $\quad \cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a$.

Exercise. Prove SSS, SAS, ASA and AAA rules of congruence of triangles (do it in two ways: with sine/cosine law and without).

Example. Use sine law to compute length of circle of radius r: $l(r) = 2\pi\sinh r$.

Corollary. Uniform statement for sine law in S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$:

$$\frac{l(a)}{\sin\alpha} = \frac{l(b)}{\sin\beta} = \frac{l(c)}{\sin\gamma},$$

where $l(r)$ is the length of circle of radius r in the corresponding geometry.

Remark. In the hyperbolic geometry, the circle length $l(r)$ grows exponentially when $r \to \infty$.

6.4 Area of hyperbolic triangle

Theorem 6.26. $S_{\triangle ABC} = \pi - (\alpha + \beta + \gamma)$.

Corollary 6.27. Area of an n-gon: $S_n = (n-2)\pi - \sum_{i=1}^n \alpha_i$.

Example. Area of hyperbolic disc of radius r is $4\pi\sinh^2(\frac{r}{2})$.