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Geometry III/IV, Term 2 (Section 7)

7 Other (projective) models of hyperbolic geometry

7.1 Klein disc revisited

Reminder: lines are represented by chords, distance in Klein disc $d(A,B) = \frac{1}{2}\left|\ln\left[A,B,X,Y\right]\right|$,
isometries are projective maps preserving the disc.

Theorem 7.1. Geometry of the Klein disc coincides with geometry of the Poincaré disc.

Remark. Hemisphere model can be projected to Klein disc, Poincare disc and upper half-plane.

Remark. When to use the Klein disc model? Working with lines and right angles.

Example. Construction of perpendicular lines, common perpendicular to two ultra-parallel lines.

7.2 The model in two-sheet hyperboloid

Define a quadratic from in $\mathbb{R}^3$ by $\boldsymbol{x}^2 = x_1^2 + x_2^2 - x_3^2$, $\boldsymbol{x} = (x_1, x_2, x_3)$, and consider the hyperboloid $H = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x}^2 = -1,\ x_3 > 0\}$.

Model: $\mathbb{H}^2 = H$ (alternatively, lines through O);
$\partial\mathbb{H}^2 = \{$lines spanning the cone $\boldsymbol{x}^2 = 0$ $\}$;
lines in $\mathbb{H}^2$: intersections of planes through O with H;
distance: $d(A,B) = \frac{1}{2}\left|\ln\left[A,B,X,Y\right]\right|$ cross-ratio of four lines in $\mathbb{R}^3$;
isometries: projective transformations preserving the quadratic from (and thus the cone).

Theorem 7.2. This determines the same hyperbolic geometry as the Klein model.

Given a quadratic from, one can define a symmetric bilinear form by

$$(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}((\boldsymbol{x}+\boldsymbol{y})^2 - \boldsymbol{x}^2 - \boldsymbol{y}^2),$$

which in our case gives

$$(\boldsymbol{x}, \boldsymbol{y}) = x_1y_1 + x_2y_2 - x_3y_3.$$

Then
- points of the $\mathbb{H}^2$: $\boldsymbol{x}^2 = (\boldsymbol{x}, \boldsymbol{x}) = -1$;
- points of the $\partial\mathbb{H}^2$: $\boldsymbol{x}^2 = (\boldsymbol{x}, \boldsymbol{x}) = 0$;
- hyperbolic line $l_{\boldsymbol{a}}$: intersection of the cone with the plane $\boldsymbol{a}^\perp = \{\boldsymbol{x} \mid (\boldsymbol{a}, \boldsymbol{x}) = 0\}$, $\boldsymbol{a}^2 > 0$.

Exercise. If $(\boldsymbol{a}, \boldsymbol{a}) > 0$ then $\boldsymbol{a}^\perp$ intersects the cone (and thus gives a hyperbolic line $l_{\boldsymbol{a}}$);
if $(\boldsymbol{a}, \boldsymbol{a}) = 0$ then $\boldsymbol{a}^\perp$ is tangent to the cone (and thus gives a point on the absolute);
if $(\boldsymbol{a}, \boldsymbol{a}) < 0$ then $\boldsymbol{a}^\perp$ does not intersect the cone (and thus $\boldsymbol{a}$ gives a point of $\mathbb{H}^2$).

Theorem 7.3. $\cosh^2 d(\boldsymbol{u}, \boldsymbol{v}) = \frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{H}^2$, i.e. for $\boldsymbol{u}, \boldsymbol{v}$ satisfying $(\boldsymbol{u}, \boldsymbol{u}) < 0$, $(\boldsymbol{v}, \boldsymbol{v}) < 0$.

Theorem 7.4. More distance formulae in terms of $Q = \left|\frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}\right|$:
- if $(\boldsymbol{u}, \boldsymbol{u}) < 0$, $(\boldsymbol{v}, \boldsymbol{v}) > 0$, then $\boldsymbol{u}$ gives a point and $\boldsymbol{v}$ gives a line $l_{\boldsymbol{v}}$ in $\mathbb{H}^2$, and $\sinh^2 d(\boldsymbol{u}, l_{\boldsymbol{v}}) = Q$;
- if $(\boldsymbol{u}, \boldsymbol{u}) > 0$, $(\boldsymbol{v}, \boldsymbol{v}) > 0$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ define two lines $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ on $\mathbb{H}^2$ and
  - if $Q < 1$, then $l_{\boldsymbol{u}}$ intersects $l_{\boldsymbol{v}}$ forming angle φ satisfying $Q = \cos^2\varphi$;
  - if $Q = 1$, then $l_{\boldsymbol{u}}$ is parallel to $l_{\boldsymbol{v}}$;
  - if $Q > 1$, then $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ are ultra-parallel lines satisfying $Q = \cosh^2 d(l_{\boldsymbol{u}}, l_{\boldsymbol{v}})$.