Durham University
Pavel Tumarkin

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Geometry III/IV, Term 2 (Section 8)

8 Isometries of $\mathbb{H}^2$

8.1 Reflections

Definition. A *reflection* r_l with respect to a hyperbolic line l is an isometry preserving the line l pointwise and swapping the half-planes.

Example. • In the Poincaré disc and upper half-plane models: reflections are represented by Euclidean reflections and inversions.
• In the Klein disc model: given A and l, one can construct $r_l(A)$.

Theorem 8.1. In hyperboloid model: given $\boldsymbol{a} \in \mathbb{R}^{2,1}$ s.t. $(\boldsymbol{a}, \boldsymbol{a}) > 0$ (i.e. $(\boldsymbol{x}, \boldsymbol{a}) = 0$ defines a line $l_{\boldsymbol{a}}$), the map $r_{\boldsymbol{a}} : \boldsymbol{x} \mapsto \boldsymbol{x} - 2\frac{(\boldsymbol{x},\boldsymbol{a})}{(\boldsymbol{a},\boldsymbol{a})}\boldsymbol{a}$ is the reflection with respect to the line $l_{\boldsymbol{a}}$.

8.2 Classification

Theorem 8.2. Any isometry of $\mathbb{H}^2$ is a composition of at most 3 reflections.

Corollary 8.3. A non-trivial orientation-preserving isometry of $\mathbb{H}^2$ has either 1 fixed point in $\mathbb{H}^2$, or 1 fixed point on the absolute, or two fixed points on the absolute.

Definition 8.4. A non-trivial orientation-preserving isometry of $\mathbb{H}^2$ is called *elliptic* if it has one fixed point in $\mathbb{H}^2$, *parabolic* if it has one fixed point in $\partial\mathbb{H}^2$, and *hyperbolic* if it has two fixed points in $\partial\mathbb{H}^2$.

Exercise. Any orientation-reversing isometry is a glide reflection, i.e. a composition of a hyperbolic isometry (preserving some line l) and a reflection in l.

Example 8.5. In the upper half-plane model, an isometry $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ is elliptic if $|d + a| < 2$, parabolic if $|d + a| = 2$ and hyperbolic if $|d + a| > 2$.

Remark 8.6 (Invariant sets of isometries)**.** Sets preserved by a given isometry for elliptic, parabolic and hyperbolic isometries are *circles*, *horocycles* and *equidistant curves* respectively.

8.3 Horocycles and Equidistant curves.

Recall the definition of a circle: a *circle* is a set of points that are on the same distance from a given point (centre).

Properties.

1. All lines through the centre are orthogonal to the circle.
2. The distance between two concentric circles γ and γ' is constant (i.e. given a point $A \in \gamma$ and a closest to A point $A' \in \gamma'$, the distance $d(A, A')$ does not depend on the choice of A).

Definition 8.7. A *horocycle* h is a limit of circles: let $P \in \mathbb{H}^2$ be a point, and l be a ray from P; for $t > 0$ let $O_t \in l$ be a point s.t. $d(P, O_t) = t$;
let $C(t)$ be a circle centred at O_t of radius t; then a horocycle $h = \lim\limits_{t\to\infty} C(t)$.
The point $X = \lim\limits_{t\to\infty} O(t) \in \partial\mathbb{H}^2$ is called the *centre* of the horocycle h.

Remark. In the Poincaré disc, every circle tangent to the absolute represents some horocycle.

Properties.

1. All lines through the centre of the horocycle are orthogonal to the horocycle.
2. The distance between two concentric horocycles h and h' is constant (i.e. given a point $A \in h$ and a closest to A point $A' \in h'$, the distance $d(A, A')$ does not depend on the choice of A).

Remark. In projective models, an equation of a circle with centre $\boldsymbol{u}$ (where $(\boldsymbol{u}, \boldsymbol{u}) < 0$) can be written as $(\boldsymbol{x}, \boldsymbol{u})^2 = c(\boldsymbol{x}, \boldsymbol{x})$ for some constant c. Similarly, an equation of a horocycle with centre $\boldsymbol{u}_0$ (where $\boldsymbol{u}_0, \boldsymbol{u}_0) = 0$) can be written as $(\boldsymbol{x}, \boldsymbol{u}_0)^2 = c(\boldsymbol{x}, \boldsymbol{x})$ for some constant c.

Definition 8.8. An *equidistant curve* e to a line l is a locus of points that are on a given distance from l.

Examples. In UHP, if l is a vertical ray 0∞, then e is a union of two (Euclidean) rays from 0 making the same angle with l. If l is a half of a Euclidean circle, then e is a union of two arcs of circles making the same angle with l (same in the Poincaré disc).

Properties.

1. All lines orthogonal to l are orthogonal to any equidistant curve of l.
2. The distance between two equidistant curves to the same line is constant.

Remark.

1. For elliptic, parabolic and hyperbolic isometry f of $\mathbb{H}^2$, through each point of $\mathbb{H}^2$ there is a unique invariant curve of f (circle, horocycle or equidistant curve respectively) and a unique line orthogonal to all invariant curves.
2. Representation of elliptic, parabolic and hyperbolic isometries as $r_2 \circ r_1$ is not unique: r_1 can be a reflection with respect to any line from the orthogonal family, by if r_1 is fixed then r_2 is unique.