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**Geometry III/IV, Term 2 (Section 9)**

# 9 Geometry in modern mathematics (some topics)

(NON-Examinable Section)

## 9.1 Taming infinity via horocycles

The idea is to choose a "ground level" for comparing infinite distances.

We illustrate this with horocycles:

- any point of a horocycle $h$ is on infinite distance from the centre $X$ of the horocycle;
- two concentric horocycles are on a (constant) finite distance from each other;
- choose "level zero" horocycle, and measure the (signed) distance to it.

**Example** (Penner, 80's)**.** Given $X, Y \in \partial H^2$, choose horocycles $h_X$ and $h_Y$ centred at these points. Let $l_{XY}$ be the finite portion of the line $XY$ lying outside of both $h_X$ and $h_Y$ (it is a signed length, may be zero or negative if $h_X$ intersects $h_Y$). Define *lambda-length* $\lambda_{XY} = \exp(l_{XY}/2)$.

**Properties.** If we change a horocycle $h_Y$ by another horocycle $h'_Y$, then any $\lambda$-length $\lambda_{ZY}$ is multiplied by a constant factor $\exp(d/2)$, where $d$ is the distance between $h_Y$ and $h'_Y$.

Recall that in Euclidean geometry we have a Ptolemy Theorem:

**Theorem** (Ptolemy Theorem)**.** In $\mathbb{E}^2$, a cyclic quadrilateral $ABCD$ satisfies

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |AD| \cdot |BC|.$$

**Theorem** (Hyperbolic Ptolemy Theorem)**.** For an ideal quadrilateral $ABCD$ (i.e. $A, B, C, D \in \partial\mathbb{H}^2$) choose any horocycles centred at $A, B, C, D$. Then

$$\lambda_{AC} \cdot \lambda_{BD} = \lambda_{AB} \cdot \lambda_{CD} + \lambda_{AD} \cdot \lambda_{BC}.$$

**Remark.** The identity does not depend on the choice of the horocycles: if we change one horocycle, all terms are multiplied by the same factor. Placing the vertices of $ABCD$ in the UHP to 0, 1, $X$ and $\infty$ respectively, and choosing appropriate horocycles ($h_D = \{y = 1\}$, others are Euclidean circles of diameter 1), the equality reduces to $\lambda_{AC} = 1 + \lambda_{BC}$, which is an easy exercise in UHP (do it!).

**Exercise.** Given an ideal triangle $A_1A_2A_3$ and $c_{12}, c_{23}, c_{31} \in \mathbb{R}_{>0}$ there exists a unique choice of horocycles centred at $A_1, A_2, A_3$ such that $\lambda_{A_iA_j} = c_{ij}$.

**Remark.** This allows to introduce coordinates on the space of hyperbolic metrics on polygons and, more generally, on triangulated surfaces; this is heavily used, in particular, in Teichmüller theory and the theory of *cluster algebras*.

## 9.2 Three metric geometries: $S^2$, $\mathbb{E}^2$, $\mathbb{H}^2$, unified

$S^2$: $d(A,B) = r\varphi$. (is tending to $\mathbb{E}^2$ when $r \to \infty$).
$\mathbb{H}^2$: $d(A,B) = R|\ln[A,B,X,Y]|$ (is tending to $\mathbb{E}^2$ when $R \to \infty$).
We want to show that $d(A,B) = \pm\frac{r}{2i}|\ln[A,B,X,Y]|$ for $S^2$.
(In case of $\mathbb{H}^2$ we consider the hyperboloid as a sphere $x_1^2 + x_2^2 + x_3^2 = -R^2$ of imaginary radius $iR$, rewriting this for $x_3' = ix_3$ we get exactly the hyperboloid model.)
To find the points $X, Y$ we use the same rule as in the hyperboloid model: $\{X,Y\} = \Pi_{AB} \cap \{(\boldsymbol{x},\boldsymbol{x})=0\}$.
- Here, the plane through $A = (a_1,a_2,a_3), B = (b_1,b_2,b_3)$ is $(a_1 + \lambda b_1, a_2 + \lambda b_2, a_3 + \lambda b_3)$.
- Intersection with the cone $(\boldsymbol{x},\boldsymbol{x}) = 0$ gives $(a_1 + \lambda b_1)^2 + (a_2 + \lambda b_2)^2 + (a_3 + \lambda b_3)^2 = 0$.
- Taking into account $(A,A) = r^2 = (B,B)$ and $(A,B) = r^2 \cos\varphi$ this gives $1 + 2\lambda\cos\varphi + \lambda^2 = 0$.
- Solving for $\lambda$ we get $X$ and $Y$: $\lambda_{1,2} = -\cos\varphi \pm i\sin\varphi$;
- $[A,B,X,Y] = [0,\infty,\lambda_1,\lambda_2] = \exp(\mp 2i\varphi)$, i.e. $\varphi = \mp\frac{r}{2i}|\ln[A,B,X,Y]|$, and $d(A,B) = \pm\frac{r}{2i}|\ln[A,B,X,Y]|$.

**Remark.** This explains the appearance of similar formulae in spherical and hyperbolic geometries; in particular, this gives a proof of the second cosine law in the hyperbolic case.

**Theorem 9.1** (Comparison Theorem by Aleksandrov-Toponogov)**.** Given $a,b,c \in R_{>0}$ such that $a+b<c$, $a+c<b$ and $b+c<a$, consider triangles in $\mathbb{H}^2, \mathbb{E}^2$ and $S^2$ with sides $a,b,c$. Let $m_{\mathbb{H}^2}$, $m_{\mathbb{E}^2}$ and $m_{S^2}$ be the medians connecting $C$ with the midpoint of $AB$ in each of the three triangles. Then $m_{\mathbb{H}^2} < m_{\mathbb{E}^2} < m_{S^2}$.

## 9.3 Discrete groups of isometries of $\mathbb{H}^2$: Examples

**Idea:** Tessellation by polygons (copies of $F$) $\to$ side pairings ($\forall a_i \in F$ there is $g_i : a_i \in gF$) $\to$
oriented graph $\Gamma$: vertices of $\Gamma$ $\leftrightarrow$ vertices $A_i$ of $F$,
edges of $\gamma$ $\leftrightarrow$ side pairings: $A_i$–$A_j$ if $g_i(A_i) = A_j$ $\to$
$\Gamma$ is a union of cycles, vertices in one cycles are called equivalent.

**Lemma.** Let $A_1,\dots,A_k$ make one cycle, so that $g_i(A_i) = A_{i+1}$, $g_k(A_k) = A_1$, where $g_i$ are side pairings of $F$ and $A_1,\dots,A_k \in \mathbb{H}^2$ (but not $\partial\mathbb{H}^2$). Then $g = g_k g_{k_1} \dots g_1$ is a rotation about $A_1$ by the angle $\alpha_1 + \dots + \alpha_k$, where $\alpha_i$ is the angle of $F$ at $A_i$.

**Claim.** Polygons $g_kF$, $g_kg_{k-1}F$,. . . ,$g_kg_{k-1}\dots g_1F$ have a common vertex $A_1$ with angles $\alpha_k, \alpha_{k-1}, \dots, \alpha_1$ at $A_1$.

**Corollary.** Elements of the group $\langle g_1,\dots,g_n\rangle$ generated by side pairings tile the neighbourhood of $A_1$ iff $\alpha_1 + \dots \alpha_k = 2\pi/m$ for $m \in \mathbb{N}$.

This necessary condition is also sufficient:

**Theorem** (Poincaré Theorem)**.** Let $F \subset \mathbb{H}^2$ be a convex polygon with finitely many sides and no ideal vertices, s.t.

(a) its sides are paired by orientation preserving isometries $\{g_1,\dots,g_n\}$;

(b) angle sum in equivalent vertices is $2\pi/m_i$ for $m_i \in \mathbb{N}$.

Then the group $G = \langle g_1,\dots,g_n\rangle$ is discrete, $F$ is its fundamental domain, and $G$ has a presentation $G = \langle g_1,\dots,g_n \mid (h_{i,k_i}^{\pm1} h_{i,k_i-1}^{\pm1} \dots h_{i,1}^{\pm1})^{m_i} = 1\rangle$, where all $h_{i,j}$ are pairing maps $g_k$, and the relations are the vertex relations.

**Examples.** - $F$ is a regular hexagon in $\mathbb{E}^2$, $G$ generated by translations pairing the opposite sides of $P$.

- $F$ is a regular hexagon in $\mathbb{E}^2$, $G$ generated by rotations by $2\pi/3$ about three non-adjacent vertices.

- $F$ is a regular hyperbolic octagon with angles $\pi/(4m)$, $m \in \mathbb{N}$, $G$ is generated by hyperbolic isometries pairing the opposite sides of $F$.

- $F$ is a polygon all whose angles are integer submultiples of $\pi$, i.e. $\pi/m$ (called a *Coxeter polygon*), $G$ is generated by reflections with respect to the sides of $F$. In this case $G = \langle r_1, \dots, r_n \mid r_i^2 = (r_i r_j)^{m_{ij}} = 1 \rangle$ is a *Coxeter group.*

- More generally, in $\mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n$ one can consider *Coxeter polytopes* as convex polytopes with all dihedral angles of type $\pi/m$. These give rise to discrete groups of isometries generated by reflections. Such polytopes in $\mathbb{E}^n, \mathbb{S}^n$ are classified by Coxeter (1934), though the hyperbolic ones are not yet classified.

## 9.4 Hyperbolic surfaces

**Definition.** A *hyperbolic surface* $S$ is a surface s.t. every point $p \in S$ has a neighbourhood isometric to a disc in $\mathbb{H}^2$.

How to construct?

### 9.4.1 Glue from hyperbolic polygons

**Examples.** Euclidean torus glued from a square with identified opposite sides; hyperbolic surface of genus 2 glued from a regular octagon with angles $\pi/4$ (opposite sides identified).

### 9.4.2 Pants decompositions (Hatcher, Thurston)

A *pair of pants* is a sphere with three holes. A hyperbolic pair of pants may be glued from two congruent right-angled hyperbolic hexagons. Gluing several pairs of pants along the boundaries, one can get (almost) every compact topological surface. Exceptions are a sphere and a torus which naturally carry spherical and Euclidean geometry respectively (but not the hyperbolic one).

### 9.4.3 Quotient of $\mathbb{H}^2$ by a discrete group

Let $G : \mathbb{H}^2$ be a discrete group action by isometries. Consider an orbit space $\mathbb{H}^2/G$. Sometimes we get a hyperbolic surface, but not always.

**Example.** A regular hyperbolic quadrilateral with angles $\pi/4$ and opposite sides identified gives a torus with a cone point (angle $\pi$ around the image of the vertices). It is not a manifold (this structure is called an orbifold).

### 9.4.4 Uniformisation theorem

A closed oriented surface is a quotient of $\mathbb{H}^2$ (or $\mathbb{E}^2$, or $S^2$) by an action of a discrete group of isometries without fixed points.

## 9.5 Review via 3D

### 9.5.1 Four models of $\mathbb{H}^3$

#### 9.5.1.1 Upper half-space

Space: $\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\}$.
Absolute: $\partial\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t = 0\}$.
Hyperbolic lines: vertical rays and half-circles orthogonal to the absolute.

Hyperbolic planes: vertical (Euclidean) half-planes and half-spheres centred at the absolute.
$d(A,B) = |ln[A,B,X,Y]|$ ($X, Y$ the ends of the line, cross-ratio computed in a vertical plane).
$\cosh d(u,v) = 1 + \frac{|u-v|^2}{2u_3v_3}$.
Isometries.
Example: Hyperbolic reflections = (Euclidean) reflections with respect to the vertical planes and Inversions with respect to the spheres centred at the absolute.

- $f \in Isom(\mathbb{H}^3)$ is determined by its restriction to the absolute.
- $Isom(\mathbb{H}^3)$ is generated by reflections (every isometry is a composition of at most 4).
- Restrictions to $\partial\mathbb{H}^3$ are compositions of (Euclidean) reflections and inversions.
- $Isom^+\mathbb{H}^3 = M\ddot{o}b$.

Spheres: Euclidean spheres (another centre).
Horospheres (limits of spheres): horizontal planes and spheres tangent to the absolute.
Equidistant (to a line): vertical cone (or banana for “half-circle” lines).
Equidistant (to a plane $\Pi$): two (Eucl) planes at the same angle to a vertical plane (at $\Pi \cap \partial\mathbb{H}^3$) or two pieces of spheres at the same angle to the sphere representing $\Pi$.

#### 9.5.1.2 Poincaré ball

Obtained by inversion from the upper half-space model.
Space: $\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 < 1\}$.
Absolute: $\partial\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 = 1\}$.
Hyperbolic lines: parts of lines and circles orthogonal to $\partial\mathbb{H}^3$.
Hyperbolic planes: parts of planes and spheres orthogonal to $\partial\mathbb{H}^3$.
$d(A,B) = |\ln[A,B,X,Y]|$ ($X, Y$ the ends of the line, cross-ratio computed in a plane).

Both Poincaré models are conformal: hyperbolic angles are represented by Euclidean angles of the same size.

#### 9.5.1.3 Klein model

Space: $\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 < 1\}$.
Absolute: $\partial\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 = 1\}$.
Hyperbolic lines: chords.
Hyperbolic planes: intersections with Euclidean planes.
$d(A,B) = \frac{1}{2}|\ln[A,B,X,Y]|$ ($X, Y$ the ends of the line).
Angles are distorted (except ones at the centre).
Right angles are easy to control.

#### 9.5.1.4 Hyperboloid model

Hyperboloid: $x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1$, $\boldsymbol{x} \in \mathbb{R}^4$.
Pseudo-scalar product: $(\boldsymbol{x},\boldsymbol{y}) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$.
Space: $(\boldsymbol{x},\boldsymbol{x}) = -1$.
Absolute: $(\boldsymbol{x},\boldsymbol{x}) = 0$.
Hyperbolic planes: $(\boldsymbol{x},\boldsymbol{a}) = 0$ for $\boldsymbol{a}$ s.t. $(\boldsymbol{a},\boldsymbol{a}) > 0$.
$d(A,B) = \frac{1}{2}|\ln[A,B,X,Y]|$ (cross-ratio of four lines).
$\cosh^2(d(pt_1, pt_2) = Q(pt_1, pt_2)$ where $Q(\boldsymbol{u},\boldsymbol{v}) = \frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}$.

#### 9.5.1.5 Orientation-preserving isometries of $\mathbb{H}^3$

In the upper half-space, orientation-preserving isometries correspond to Möbius transformations of $\partial\mathbb{H}^3$: $z \mapsto \frac{az+b}{cz+d}$ with $z \in \partial\mathbb{H}^3$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$.
- Parabolic: 1 fixed point on $\partial\mathbb{H}^3$, conjugate to $z \mapsto z + 1$.
- Non-parabolic: 2 fixed points on $\partial\mathbb{H}^3$, conjugate to $z \mapsto az$.
elliptic, $|a| = 1$, rotation about a vertical line.
hyperbolic, $a \in \mathbb{R}_{>0}$, (Euclidean) dilation.
loxodromic, (otherwise), composition of rotation and dilation.

## 9.6 Some polytopes in $\mathbb{H}^3$

- Ideal tetrahedron. It is not unique up to isometry (there are 2 parameters = 2 dihedral angles).
- Regular right-angled dodecahedron.
- Regular right-angled ideal octahedron.

## 9.7 Geometric structures on 3-manifolds

Similar to surfaces from polygones, can glue 3-manifolds from polyhedra. Need to check angles around edges (Poincaré Polyhedron Theorem).

## 9.8 Geometrisation conjecture

**Conjecture** (Thurston)**.** All topological 3-manifolds are geometric manifolds, i.e. every oriented compact 3-manifold without boundary can be cut into pieces having one of the following 8 geometries: $S^3$, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $Nil$, $Sol$ and universal cover of $SL(2, \mathbb{R})$.

(1982) Thurston: proved geometrisation conjecture for Haken manifolds. (Fields medal, 1982)
In particular, closed atoroidal Haken manifolds are hyperbolic.

(2003) Perelman: general proof of the geometrisation conjecture. (Fields medal, 2006)
This also proves Poincaré conjecture:
Every simply-connected closed 3-manifold is a 3-sphere. (Clay Millennium Prize).