

## Geometry III/IV, Solutions: weeks 11–12

### Möbius transformations, inversion

11.1. Show that Möbius transformations form a group.

*Solution:*

We need to prove 4 properties of a group:

1) composition of Möbius transformations is a Möbius transformation:

$$g(f(z)) = \frac{a_2 \frac{a_1 z + b_1}{c_1 z + d_1} + b_2}{c_2 \frac{a_1 z + b_1}{c_1 z + d_1} + d_2} = \frac{(a_2 a_1 + b_2 c_1)z + (a_2 b_1 + b_2 d_1)}{(c_2 a_1 + d_2 c_1)z + (c_2 b_1 + d_2 d_1)}.$$

We also need to check the condition  $ad - bc \neq 0$  for  $g(f(z))$ . Computing  $ad - bc$  we get 8 terms, four of which cancel, and four other can be regrouped to

$$(a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2),$$

which is non-zero since the multiples are non-zero.

2) existence of an **identity** map in the set of Möbius transformations:

$$f_{id}(z) = \frac{z + 0}{0 \cdot z + 1} = z$$

3) existence of an **inverse** map in the set of Möbius transformations: we find the inverses for  $f_1(z) = az + b$  and  $f_2(z) = 1/z$ :

$$f_1^{-1}(z) = \frac{1}{a}z - \frac{b}{a} \quad f_2^{(-1)}(z) = \frac{1}{z}.$$

It was shown in the lecture that every Möbius transformation can be obtained as a composition of several transformations of types  $f_1$  and  $f_2$ . The inverse of the composition is the composition of inverses  $(g_k \circ \dots \circ g_1)^{-1} = g_1^{-1} \circ \dots \circ g_k^{-1}$ .

4) associativity: can be computed directly.

**Remark:** we demonstrate the properties by the direct computations, but of course one can use instead the same reasoning as in the lecture, i.e. modelling the action of  $f(z) = \frac{az+b}{cz+d}$  by multiplication by matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then all group properties of Möbius transformations follow from the corresponding group properties of  $GL(2, \mathbb{C})$ .

11.2. Find a Möbius transformation which takes 1, 2, 3 to 0, 1,  $\infty$ .

*Solution:*

Let  $f(z) = \frac{az+b}{cz+d}$ . Since  $f(1) = 0$ , we see that  $b = -a$ . Since  $f(3) = \infty$ , we have  $d = -3c$  (and we can also assume  $c = 1$  as  $c$  cannot be zero). Finally, since  $f(2) = 1$ , we have  $f(2) = \frac{a(2-1)}{2-3} = -a = 1$ . Thus,  $f(z) = \frac{-z+1}{z-3}$ . (Notice that  $ad - bc = 3 - 1 \neq 0$ , so it is a Möbius transformation.)

11.3. (★)

- (a) Let  $l$  be a line and  $\gamma$  be a circle. Show that  $\gamma$  is orthogonal to  $l$  if and only if  $l$  contains the centre of  $\gamma$ .
- (b) Let  $\gamma_1, \gamma_2, \gamma_3$  be three mutually orthogonal circles on the plane. Show that there exists a Möbius transformation which takes them to the curves  $\{x = 0\}$ ,  $\{y = 0\}$  and  $x^2 + y^2 = 1$ .

*Solution:*

- (a) A line  $l$  through the centre  $O$  of the circle is orthogonal to the circle as the radius is orthogonal to the tangent line. In more details, the reflection  $r_l$  with respect to  $l$  preserves the circle since  $O \in l$ , so, it preserves the intersection point of the circle with  $l$ , and hence, preserves the tangent at this point (by uniqueness of the tangent). This implies that the tangent is orthogonal to  $l$ .
- Now, consider a line not through  $O$  intersecting the circle at points  $A$  and  $B$ . As it was shown above, the radius  $OA$  is orthogonal to the tangent, which implies that  $BA$  is not orthogonal to it (as  $O \notin AB$ ).
- (b) First, we map (by a Möbius transformation) two intersections of two of the circles into 0 and  $\infty$  (this is possible by triple transitivity). This maps the two circles into two perpendicular lines. Then we apply a Euclidean isometry (composition of a rotation and translation, i.e. a Möbius transformation) to move these lines to the coordinate axes. The third circle is mapped to a circle or line orthogonal to both axes. This is clearly impossible for a line. For a circle this is only possible when  $O$  is the centre of the circle (here we use (a)). Finally, applying  $f(x) = kz$  with  $k \in \mathbb{R}$  if needed, we can make sure that the third curve is the unit circle.

- 11.4. (★) Let  $\gamma$  be a circle and  $P$  be a point lying outside of  $\gamma$ . Let  $l$  be a line through  $P$  intersecting  $\gamma$ , denote by  $A$  and  $B$  the intersection points of  $l$  with  $\gamma$ . Show that the product  $|PA| \cdot |PB|$  does not depend on the choice of  $l$ . (This product is also called the *power of  $P$  with respect to  $\gamma$* ).

*Solution:*

Let  $l_1 \neq l$  be another line though  $P$  and let  $A_1, B_1$  be the intersection points of  $l_1$  with  $\gamma$  (see Figure (a) below). The quadrilateral  $ABB_1A_1$  is inscribed into  $\gamma$ , so by E29,

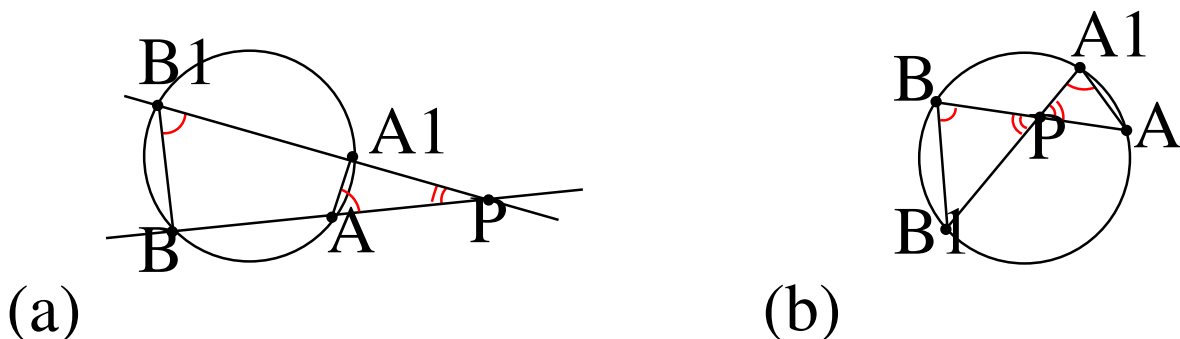
$$\angle BB_1A_1 + \angle BAA_1 = \pi,$$

which implies  $\angle BB_1A_1 = \angle A_1AP$ . Hence, the triangle  $B_1BP$  is similar to the triangle  $AA_1P$  (by two angles: they have the common angle  $P$  and a pair of equal angles as above). Therefore,

$$\frac{PA}{PB_1} = \frac{PA_1}{PB},$$

which implies

$$PA \cdot PB = PA_1 \cdot PB_1.$$



**11.5.** The same question as 11.4, but  $P$  lies inside  $\gamma$ .

*Solution:*

This is quite similar to above (see Figure (b)). This time the triangles  $B_1BP$  and  $AA_1P$  are similar again by two angles (angles at  $P$  are vertical;  $\angle B_1BP = \angle AA_1P$  as angles in the same circular segment, E28), and we get exactly the same equalities as above.

**12.1.** Prove the theorem of Ptolemy: for a cyclic quadrilateral  $ABCD$ , the following equality holds:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

*Solution:*

First, divide by  $AC \cdot BC$ , so that we need to prove

$$\frac{AB \cdot CD}{AC \cdot BD} + \frac{AD \cdot BC}{AC \cdot BD} = 1.$$

Rewriting products as ratios

$$\frac{AB}{AC} : \frac{DB}{DC} + \frac{AD}{AC} : \frac{BD}{BC} = 1, \tag{1}$$

we easily recognize (modules of) two cross-ratios  $[B, C, A, D]$  and  $[D, C, A, B]$  on the left.

Here, we understand  $A, B, C, D$  as complex numbers and  $AB$  as  $B - A$  (we have a right to do it since the points lie on a circle, see a remark in lectures). We can also see that the cross-ratios are positive: for that we can map the circle to a real line by a Möbius transformation  $f$  (which preserves cross-ratios), so that  $f(A) < f(B) < f(C) < f(D)$ ; then the positivity of the cross-ratios above easily follows.

So, we see that in the case of a cyclic quadrilateral we can understand equation (1) as  $[B, C, A, D] + [D, C, A, B] = 1$ . Now, we notice that the second cross-ratio is obtained from the first one by permutation of the points and recall from Problem 7.8 that if  $[B, C, A, D] = \lambda$  then  $[D, C, A, B] = 1 - \lambda$ .

**12.2.** (★) (Inversion with ruler and compass).

In this question, you need to find and describe algorithms for certain constructions as well as to justify them. You can use without proofs and further descriptions the following constructions:

- midpoint of a given segment;
- perpendicular bisector for a given segment;
- the line perpendicular to a given line through a given point.

This is a long question with an easy start and more complicated parts at the end. Please, try to submit parts **(b)**–**(e)**. You are welcome to submit solutions/sketches for later parts, but formally it is not the part of the written assignment.

(a) Given a circle  $\gamma$ , construct its centre.

*Solution:*

We take any three points  $A, B, C$  on the circle. Then the perpendicular bisectors of  $AB$  and  $AC$  pass through the centre of the circle, so the centre is just the intersection of two perpendicular bisectors (we know how to construct perpendicular bisectors from the problem class last term).

(b) Given segments of length  $a$  and  $b$  construct a segment of length  $h$  satisfying  $h^2 = a \cdot b$ .

*Solution:*

We use a right-angled triangle  $ABC$  (with right angle  $C$ ), its altitude  $CH$  satisfies  $CH^2 = HA \cdot HB$  (see E23).

To construct this triangle start from the points  $H$  and find the points  $A$  and  $B$  on one line (so that  $AH = a$ ,  $BH = b$  and  $H$  lies between  $A$  and  $B$ ). To construct the points  $C$ , the vertex of the right angle, draw a circle  $\gamma_1$  with diameter  $AB$  (centred at midpoint of  $AB$ ) and find  $C$  as intersection of  $\gamma_1$  with the line perpendicular to  $AB$  through  $H$ . Then  $\angle ACB$  is right as it is an angle in a semicircle (E26), and  $CH$  is the altitude by construction. So we can apply E20 to see that the segment  $CH$  satisfies all required properties.

(c) Given a circle  $\gamma$  and a point  $P$  outside the circle, construct a line  $PQ$  tangent to  $\gamma$ .

*Solution:*

Consider the tangent line  $PQ$  assuming that  $Q \in \gamma$ . Using the result of Question 11.4 we see that  $PQ^2 = PA \cdot PB$ , where  $A, B$  are the intersection points of  $\gamma$  with arbitrary line  $l$  through  $P$ . We can easily construct an arbitrary line  $l$  through  $P$  and the intersection points  $A$  and  $B$ , so we get the segments  $PA$  and  $PB$ . So, by Part (b) we can construct the segment of length  $PQ$ , so that  $Q$  is the intersection of  $\gamma$  with the circle centred at  $P$  of radius  $PQ$ .

(d) Given a circle  $\gamma$  and a point  $A$  outside the circle, construct the inversion image of  $A$

*Solution:*

First, construct a tangent line  $AQ$  (with  $Q \in \gamma$ ) to the circle  $\gamma$  (we can do it by Part (c)).

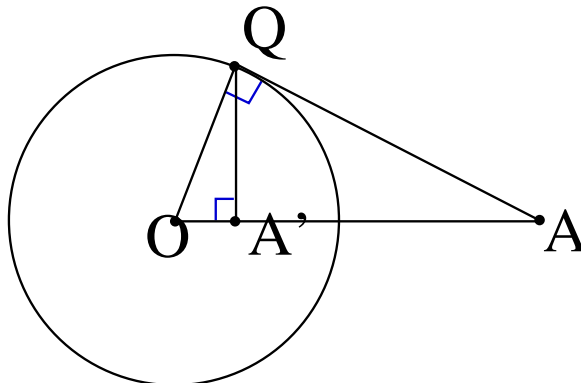
Now, let  $A'$  be the orthogonal projection of  $Q$  to  $OA$ . We will prove that  $A'$  is the inversion image of  $A$  with respect to  $\gamma$ .

Notice that we can construct  $A'$ : by Part (a) we can construct the centre  $O$  of  $\gamma$ , then we know how to construct the orthogonal projection of a given point ( $Q$ ) to a given line ( $OA$ ) since we did that in a problems class.

To prove that  $A'$  is the inversion image of  $A$  with respect to  $\gamma$ , consider the right triangles  $AOQ$  and  $QOA'$  (the angle  $\angle QA'A$  is right by construction and the angle  $\angle OQA$  is right as it is an angle between the radius and the tangent). These triangles also have a common angle  $O$ , so they are similar. Therefore,

$$\frac{OA}{OQ} = \frac{OQ}{OA'},$$

which implies  $OA \cdot OA' = R^2$  where  $R = OQ$  is the radius of  $\gamma$ .



(e) Construct the inversion image for the point  $A'$  lying inside the circle  $\gamma$ .

*Solution:*

Invert the construction in (d): first find  $Q$  as an intersection of  $\gamma$  with the line through  $A'$  perpendicular to  $OA$ , then draw the tangent line at  $Q$  (it is orthogonal to the radius  $OQ$ ) and find  $A$  as the intersection with  $OA$ .

(f) Let  $O$ ,  $A'$  and  $A$  be three points lying on a line ( $A'$  lies between  $O$  and  $A$ ).

Construct a circle  $\gamma$  centred at  $O$  such that the inversion with respect to  $\gamma$  takes  $A$  to  $A'$ .

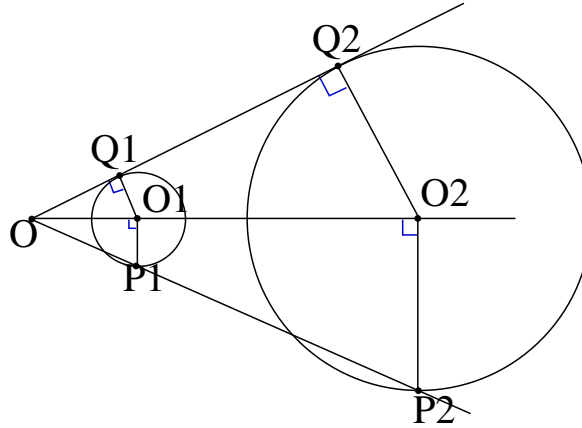
*Solution:*

This is similar to Part (b) (with  $a = OA$ ,  $b = OA'$  and the unknown radius  $R = h$ ).

(g) Given two circles  $\gamma_1$  and  $\gamma_2$ , construct a line tangent to both of them.

*Solution:*

First, suppose that the required tangent line  $l$  is already constructed. Denote by  $O_1$  and  $O_2$  the centres of the two circles, denote also  $Q_1 = l \cap \gamma_1$  and  $Q_2 = l \cap \gamma_2$ . Let  $O$  be the intersection point of  $l$  with  $O_1O_2$  (suppose also that  $l$  is not parallel to  $O_1O_2$ ). Then the circle  $\gamma_2$  can be obtained from  $\gamma_1$  by a homothety with centre  $O$  and coefficient  $OO_2/OO_1$  (indeed, the triangles  $OO_1Q_1$  and  $OO_2Q_2$  are similar by two angles as both have a right angle and have a common angle  $O$ , so  $OO_2/OO_1 = OQ_2/OQ_1$ ). In particular, if  $P_1 \in \gamma_1$  and  $P_2 \in \gamma_2$  are the farthest points of  $\gamma_1$  and  $\gamma_2$  from the line  $O_1O_2$ , then  $O$  lies on the line  $P_1P_2$ .



We can use the consideration above to construct  $O$ . Indeed, it is easy to construct the points  $P_1$  and  $P_2$  ( $P_i = \gamma_i \cap l_i$ , where  $l_i$  is the line through  $O_i$  perpendicular to  $O_1O_2$ ). Then we find  $O$  as the intersection of  $P_1P_2$  with  $O_1O_2$ . Finally, we construct  $Q_1Q_2 = OQ_1$  as in Part (c).

If the line  $P_1P_2$  is parallel to  $O_1O_2$ , then  $P_1P_2$  is the required tangent line.

(h) Given two circles  $\gamma_1$  and  $\gamma_2$  of different sizes, construct an inversion which takes  $\gamma_1$  to  $\gamma_2$  and takes  $\gamma_2$  to  $\gamma_1$ .

(You need to construct the centre and the radius of the circle of inversion).

*Solution:*

Let  $O_1$  and  $O_2$  be the centres of the circles and let  $O = O_1O_2 \cap l$  be the intersection of  $O_1O_2$  with the line  $l$  tangent to both circles (constructed as in (g)). Let  $l \cap \gamma_1 = Q_1$ ,  $l \cap \gamma_2 = Q_2$  be the intersection points. Let

$H \in O_1O_2$  be a point such that  $OH^2 = OQ_1 \cdot OQ_2$  (constructed as in (b)). Let  $\gamma$  be the circle centred at  $O$  of radius  $OH$ . We will prove that the inversion  $I_\gamma$  in  $\gamma$  swaps the circles  $\gamma_1$  and  $\gamma_2$ .

As  $OH^2 = OQ_1 \cdot OQ_2$ , we see that  $I_\gamma(Q_1) = Q_2$  and  $I_\gamma(Q_2) = Q_1$ . Furthermore,  $I_\gamma$  takes the tangent line  $Q_1Q_2$  to itself (as it passes through  $O$ ) as well as it takes to itself the other tangent line  $m$  (another line through  $O$  tangent to both circles). So,  $I_\gamma$  should take the circle  $\gamma$  to a circle tangent to  $l$  at  $Q_2$  and also tangent to  $m$  (since inversion takes circles not through the origin to circles and preserves angles, and since  $I_\gamma(Q_1) = Q_2$ ). It is easy to see that  $\gamma_2$  is the only circle satisfying these conditions, so,  $I_\gamma(\gamma_1) = \gamma_2$ . The same reasoning shows that  $I_\gamma(\gamma_2) = \gamma_1$ .

(i) Given two circles  $\gamma_1$  and  $\gamma_2$  of different sizes, find an inversion which takes them to a pair of equal circles.

(You need to construct the centre and the radius of the circle of inversion).

*Solution:*

Let  $\gamma_0$  be the circle such that the inversion with respect to  $\gamma_0$  swaps  $\gamma_1$  and  $\gamma_2$  (constructed as in (h)).

Denote by  $O$  the intersection point of  $\gamma_0$  and  $O_1O_2$  not lying between  $O_1$  and  $O_2$ , denote by  $H$  the other intersection. Consider an inversion  $I$  with respect to the circle  $\gamma$  centred at  $O$  of radius  $OH$ .

As  $\gamma_0$  passes through  $O$ ,  $I$  takes  $\gamma_0$  to a line  $l_0$ . As neither of  $\gamma_1$  and  $\gamma_2$  passes through  $O$ , both  $I(\gamma_1)$  and  $I(\gamma_2)$  are circles. Let  $r_0$  be the reflection with respect to  $l_0$ . We will show that  $r_0$  swaps the circles  $I(\gamma_1)$  and  $I(\gamma_2)$ . Then we conclude that the circles are equal.

To show that  $r$  swaps the circles  $I(\gamma_1)$  and  $I(\gamma_2)$ , consider the composition  $f = I \circ I_{\gamma_0} \circ I$ . Clearly,  $f$  takes  $l_0$  to itself pointwise ( $I$  takes a point on  $l_0$  to a point on  $\gamma_0$ , then  $I_{\gamma_0}$  preserves it and  $I$  takes it back to the initial place). Also, as a composition of inversions, it should take lines and circles to lines and circles. Since the infinite point is preserved by  $f$  (lying on  $l_0$ ),  $f$  takes lines to lines. So, by Theorem 3.7. (Fundamental Theorem of affine geometry)  $f$  is an affine map. Also, as a composition of inversions,  $f$  preserves angles. So,  $f$  is a similarity map. Furthermore, as  $f$  preserves all points of  $l_0$ ,  $f$  is an isometry. Finally, as  $f$  changes the orientation,  $f$  is the reflection  $r_0$  with respect to the line  $l_0$ :

$$r_0 = I \circ I_{\gamma_0} \circ I.$$

Now,

$$r_0(I(\gamma_1)) = I \circ I_{\gamma_0} \circ I(I(\gamma_1)) = I \circ I_{\gamma_0}(\gamma_1) = I(\gamma_2),$$

and similarly,

$$r_0(I(\gamma_2)) = I(\gamma_1).$$

So, the circles  $I(\gamma_1)$  and  $I(\gamma_2)$  are of the same size.

### 12.3. What type is the transformation $1/z$ ?

(Hint: parabolic or not? if not, then is it elliptic, or hyperbolic, or loxodromic?)

*Solution:*

The transformation  $f(z) = 1/z$  fixes the points  $z = 1$  and  $z = -1$ , so  $f$  is not parabolic.  $f^2 = Id$ , so the fixed points are not attracting or repelling. So,  $1/z$  is elliptic.

### 12.4. Write the following transformations as compositions of reflections/inversions:

(a)  $2z$       (b)  $-z$       (c)  $z + 1$       (d)  $\frac{1}{z}$

*Solution:*

We will present each of the transformations as a composition of two inversions/reflections  $f(z) = f_2 \circ f_1(z)$ . Denote by  $s_1$  and  $s_2$  the fixed set (i.e. circle or line) of  $f_1$  and  $f_2$  respectively.

(a)  $s_1$  and  $s_2$  are circles  $|z| = 1$  and  $|z| = \sqrt{2}$ ;  $f_1(z) = 1/\bar{z}$ ,  $f_2(z) = 2/\bar{z}$ .

(b)  $s_1$  and  $s_2$  are lines  $\text{Im}(z) = 0$  and  $\text{Re}(z) = 0$ ;  $f_1(z) = \bar{z}$ ,  $f_2(z) = -\bar{z}$ .

(c)  $s_1$  and  $s_2$  are line  $\text{Re}(z) = 0$  and  $\text{Re}(z) = 1/2$ ;  $f_1(z) = -\bar{z}$ ,  $f_2(z) = \overline{-z - \frac{1}{2}} + \frac{1}{2} = -\bar{z} + 1$ .

(d)  $s_1$  is the real line,  $s_2$  is the unit circle.  $f_1 = \bar{z}$ ,  $f_2 = \frac{1}{\bar{z}}$ .

**12.5.** Let  $I$  be the inversion with respect to the unit circle  $|z| = 1$ . Find the image  $I(l)$  of the line  $l$  given by the equation  $\text{Re } z = 2$ .

*Solution:*

$l$  passes through infinity and does not pass through the origin, so  $I(l)$  is a circle through the origin.  $l$  passes through  $z = 2$ , so  $I(l)$  passes through  $z = 1/2$ . Both  $l$  and the unit circle are symmetric with respect to the line  $y = 0$ , so,  $I(l)$  is symmetric with respect to the line  $y = 0$ . (In other words,  $I$  preserves this line and takes the line orthogonal to it to a circle orthogonal to it). Hence,  $I(l)$  is the circle  $(x - \frac{1}{4})^2 + y^2 = \frac{1}{16}$ .

**12.6.** Do the points  $-1 - 2i$ ,  $-1 + 2i$ ,  $3 + i$ ,  $3 - i$  lie on one line or circle?

*Solution:*

Four points lie on the same line or circle if their cross-ratio is real.

$$[-1 - 2i, -1 + 2i, 3 + i, 3 - i] = \frac{(3+i+1+2i)}{3+i-(-1+2i)} / \frac{3-i+1+2i}{3-i-(-1+2i)} = \frac{4+3i}{4-i} \frac{4-3i}{4+i} \in \mathbb{R}$$

Hence, these points lie on one circle or line (it is easy to see that it is actually a circle).

**12.7.** Show that a finite order Möbius transformation is elliptic.

( $g$  is called *of finite order* if  $g^n = id$  for some integer  $n$ ).

*Solution:*

First, notice that a conjugation preserves the order of the transformation.

A parabolic Möbius transformation is conjugate to  $z + 1$ . This transformation is of infinite order, so a finite order Möbius transformation is not parabolic.

A non-parabolic Möbius transformation is conjugate to  $\alpha z$ ,  $\alpha \in \mathbb{C}$ ,  $\alpha \neq 0$ . If  $|\alpha| \neq 1$  then  $|z^n|$  growth (or decreases) when  $n$  tends to infinity. So for a finite order Möbius transformation we have  $|\alpha| = 1$ . Hence, it is elliptic.

**12.8.** Find a parabolic Möbius transformation preserving the point  $z = 1$ .

*Solution:*

Let  $f(z) = z + 1$ , it is a parabolic transformation preserving  $\infty$ . Let  $g$  be a transformation which takes 1 to  $\infty$ , say  $g = \frac{1}{z-1}$ . Then  $\varphi = g^{-1} \circ f \circ g$  preserves  $z = 1$ .

To find  $g^{-1}$  notice that  $g = g_2 \circ g_1$  where  $g_1(z) = z - 1$ ,  $g_2(z) = 1/z$ . Since  $g_1^{-1} = z + 1$ ,  $g_2^{-1} = 1/z$ , we have  $g^{-1} = g_1^{-1} \circ g_2^{-1} = \frac{1}{z} + 1$ .

Hence,  $\varphi = g^{-1} \circ f \circ g = \frac{1}{\frac{1}{z-1} + 1} = \frac{z-1}{1+z-1} + 1 = \frac{z-1}{z} + 1 = \frac{-1}{z} + 2$  is a parabolic transformation preserving  $z = 1$ .

**12.9.** Find a Möbius transformation mapping the disc  $|z| < 1$  to the half-plane  $\text{Re } z > 2$ .

*Solution:*

First, we will find a transformation  $f(z) = \frac{az+b}{cz+d}$  mapping  $1, i, -1$  to  $0, 1, \infty$ :

Since  $f(1) = 0$  we have  $a + b = 0$ , so  $b = -a$ .

Since  $f(-1) = \infty$ , we have  $-c + d = 0$ , so  $d = c$ .

Since  $f(i) = 1$ , we have  $\frac{ai-a}{ci+c} = 1$ , which implies

$$\frac{a}{c} = \frac{i+1}{i-1} = \frac{(i+1)^2}{(i-1)(i+1)} = \frac{-1+2i+1}{-2} = -i,$$

so,  $c = ai$ .

Hence,  $f(z) = \frac{az-a}{aiz+ai} = -i\frac{z-1}{z+1}$ .

Since  $f$  takes  $1, i, -1$  to  $0, 1, \infty$ , it takes the unit circle to  $\mathbb{R}$ .

Since  $f(0) = -i(-1) = i$ , the disc is mapped to the upper half-plane.

Now, apply a rotation  $g$  by  $\pi/2$  clockwise around  $z = 2$ :  $g(z) = (z - 2)(-i) + 2 = -iz + (2 + 2i)$ . Taking a composition  $g \circ f : z \mapsto \frac{z(1+2i)+(3+2i)}{z+1}$  we obtain the required map.

**12.10.** Let  $I_0$  be the inversion with respect to the circle  $|z| = 1$ , and  $I_1$  be the inversion with respect to the circle  $|z - 1| = 1$ . What is the type of the Möbius transformation obtained as a composition  $I_1 \circ I_0$ ?

(Hint: try to find a geometric solution without writing any formulae).

*Solution:*

The circles  $|z| = 1$  and  $|z - 1| = 1$  have two points of intersection. Both of these two points are fixed points of  $f$ . So,  $f$  is not parabolic.

Let us check whether the fixpoints of  $f$  are repellent/attractive. Consider the orbit of the point  $\infty$ :

$$\infty \xrightarrow{I_0} 0 \xrightarrow{I_1} 0 \xrightarrow{I_0} \infty \xrightarrow{I_1} 1 \xrightarrow{I_0} 1 \xrightarrow{I_1} \infty$$

This implies that  $(I_1 \circ I_0)^n(\infty)$  is  $0, 1$  or  $\infty$  for any  $n$ , so that it does not tend to any of the fixed points of  $f$ . Hence, the fixed points are not attractive or repellent. So,  $f = I_1 \circ I_0$  is an elliptic transformation.

**Remark:** we can also see that  $f$  is of order 3, since  $f^3$  preserves  $\infty$  and two intersection points of the circles.

## References:

1. Lectures (Möbius transformations, Inversion, Stereographic projection) and Lecture V in Prasolov's book.
2. Section 1.1 of the book *Hyperbolic Geometry* by Caroline Series.
3. For introduction and discussion of inversion you can check the following sources (both linked from the course on-line resources page):
  - *Circle inversion* by Malin Christersson.
  - *Inversion in a circle* by Tom Davis.