Durham University
Pavel Tumarkin

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Geometry III/IV, Solutions: weeks 13–14

Hyperbolic geometry: conformal models

13.1. Draw in each of the two conformal models (Poincaré disc and upper half-plane):

(a) two intersecting lines;

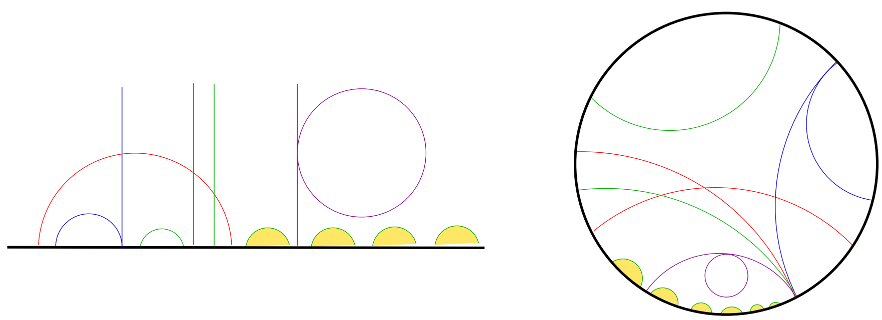
(b) two parallel lines;

(c) two ultra-parallel lines;

(d) infinitely many disjoint half-planes;

(e) a circle tangent to a line.

Solution: One solution is indicated by colors: (a), (b), (c), (d), (e).



13.2. In the upper half-plane model draw

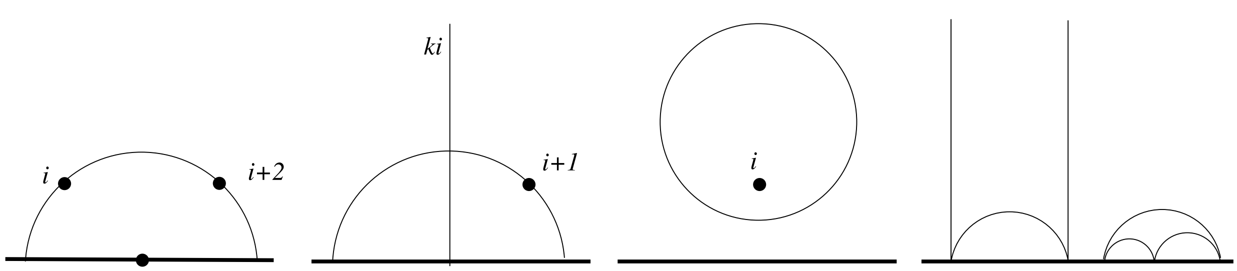
(a) a line through the points i and $i+2$;

(b) a line through $i+1$ orthogonal to the line represented by the ray $\{ki \mid k>0\}$;

(c) a circle centred at i (just sketch it, no formula needed);

(d) a triangle with all three vertices at the absolute (such a triangle is called *ideal*).

Solution:



13.3. Prove SSS, ASA and SAS theorems of congruence of hyperbolic triangles.

Solution:

1. SSS: Let $A_1B_1C_1$ and $A_2B_2C_2$ be two hyperbolic triangles satisfying SSS (i.e. having the same side lengths). First we apply an isometry which takes A_2 to A_1 and B_2 to B_1. Then the points C_1 and C_2 lie on the intersection of (hyperbolic) circles γ_A (centred at A_1 of radius A_1C_1) and γ_B (centred at B_1 of radius B_1C_1). Consider these circles in Poincaré disc or half-plane model. Since hyperbolic circles are represented by Euclidean circles, two circles have at most two intersection points. Moreover, the two intersection points are symmetric with respect to the (hyperbolic) line A_1B_1 Hence, there is an isometry which takes $A_2B_2C_2$ to $A_1B_1C_1$.

2. ASA: Suppose that $A_1B_1 = A_2B_2$ $\angle A_1 = \angle A_2$ and $\angle B_1 = \angle B_2$. Apply an isometry which takes A_2 to A_1 and B_2 to B_1. Then C_1 and C_2 lie on a (hyperbolic) ray starting from A_1 and making angle $\angle A_1$ with A_1B_1. There exists exactly one such ray in each half-plane with respect to the line A_1B_1 (this is especially clear if A_1 is the centre of the Poincaré disc model). Similarly, C_1 and C_2 lie on a (hyperbolic) ray starting from B_1 and making angle $\angle B_1$ with A_1B_1. As two rays have at most one intersection, we get at most one candidate for the point C_1 and C_2 in each of two half-planes, also these two candidates are symmetric with respect to A_1B_1. Hence, $A_2B_2C_2$ can be transformed to $A_1B_1C_1$ by an isometry.

3. SAS: Suppose that $A_1B_1 = A_2B_2$, $\angle A_1 = \angle A_2$ and $A_1C_1 = A_2C_2$. First, map the angle $\angle A_2$ to $\angle A_1$, then the points C_1 and C_2 lie on the given distances on the given lines.

13.4. Let ABC be a triangle. Let $B_1 \in AB$ and $C_1 \in AC$ be two points such that $\angle AB_1C_1 = \angle ABC$. Show that $\angle AC_1B_1 > \angle ACB$.

Solution:

Consider the quadrilateral B_1BCC_1. If $\angle AC_1B_1 \leq \angle ACB$ then the sum of angles of B_1BCC_1 is greater or equal to 2π. On the other hand, we can divide B_1BCC_1 by a diagonal into two triangles, each having a sum of angles less than π. The contradiction shows that $\angle AC_1B_1 > \angle ACB$.

13.5. Show that there is no "rectangle" in hyperbolic geometry (i.e. no quadrilateral has four right angles).

Solution:

Suppose there is a rectangle $ABCD$. Then its sum of angles is 2π. Decompose it into two triangles by a diagonal AC. The sum of angles of ABC is smaller than π, the sum of angles of ACD is smaller than π but the sum of these two sums of angles equal to the sum of angles of $ABCD$. Contradiction.

13.6. ($\star$) Given an acute-angled polygon P (i.e. a polygon with all angles smaller or equal to $\pi/2$) and lines m and l containing two disjoint sides of P, show that l and m are ultra-parallel.

Solution:

Let $A_1A_2 \dots A_n$ be an acute-angled n-gon.

First, let us prove that the lines containing two "almost adjacent" rays A_1A_2 and A_4A_3 are disjoint: i.e. suppose $B = A_1A_2 \cap A_4A_3$. Then the triangle A_2A_3B has at least two non-acute angles, which contradicts the fact the angle sum of a hyperbolic triangle is less than π.

Similarly, assuming that the rays A_1A_2 and A_5A_4 do intersect, we see a (non-convex) quadrilateral, with two non-acute angles and one angle bigger than π, which again contradicts the angle sum.

In general, intersection of the rays A_1A_2 and A_kA_{k-1} will result in an $(k-1)$-gon breaking the angle sum theorem (one can use induction to show that).

14.1. Given non-negative real numbers α, β, γ such that $\alpha+\beta+\gamma < \pi$, show that there exists a hyperbolic triangle with angles α, β, γ.

Solution:

Put a vertex A of angle α in the centre of the Poincaré disc model. Let AX and AY be the rays forming the angle. Let T be a point on AX. Let TZ be a ray emanating from T and such that $\angle ZTA = \beta$. Consider the point $C(T) = TZ \cap AY$. When T is very close to A the hyperbolic line TZ is very close to a Euclidean line through the same points, so the hyperbolic sum of angles of the triangle $\triangle ATC(T)$ is very close to π (and thus the angle $\angle AC(T)T$ is close to $\pi - (\alpha + \beta)$. As T runs away from A to Y, the angle $\angle AC(T)T$ monotonically decreases to zero (when TZ is parallel to AY). Since $0 \le \gamma < \pi - (\alpha+\beta)$, for some intermediate point T_0 the angle $\angle AC(T_0)T_0$ will be equal to γ, so $\triangle AT_0C(T_0)$ is the required triangle.

14.2. Show that there exists a hyperbolic pentagon with five right angles.

Solution:

Consider a Euclidean regular pentagon P_{Eucl}. Draw P_{Eucl} so that the centre of P_{Eucl} coincides with the centre O of Poincaré disc model. Consider the hyperbolic pentagon P spanned by the vertices of P_{Eucl}. When P_{Eucl} is very small (but still centred at O) the angle of P are almost the same as the angles of P_{Eucl}. When P_{Eucl} is inscribed into the absolute the angles of P are equal to zero. Notice that the angles of P_{Eucl} are obtuse (more precisely, they are equal to $3\pi/5$). So, by continuity we see that there exists some intermediate size of P_{Eucl} such that the corresponding hyperbolic regular pentagon has right angles.

14.3. An *ideal* triangle is a hyperbolic triangle with all three vertices on the absolute.

(a) Show that all ideal triangles are congruent.

(b) Show that the altitudes of an ideal triangle are concurrent.

(c) Show that an ideal triangle has an inscribed circle.

Solution:

(a) There exists a hyperbolic isometry which takes any given triple of points on the absolute to any other triple. So, it takes a triangle spanned by the given three points to the triangle spanned by the other three points.

(b) Part (a) implies that any ideal triangle can be represented by a "regular" ideal triangle in the Poincaré disc model (i.e. by a triangle with vertices $1, e^{i\pi/3}, e^{2i\pi/3}$). The symmetry shows that all altitudes of this triangle pass through the centre O of the model.

(c) Similarly to part (b), looking at the "regular" representative, we can see that there is an inscribed circle centred at O (we can take a very small circle and start to blow it up till it will touch one of the sides; by symmetry reasons it will touch all other sides at the same time).

14.4. It was proved in lectures that an isometry fixing 3 points of the absolute is the identity map. How many isometries fix two points of the absolute? Classify the isometries fixing 0 and ∞ in the upper half-plane model.

Solution:

We will work in the upper half-plane model. Let f be an isometry fixing two points of the absolute. First, we can conjugate f by an isometry h which takes the fixpoints of f to 0 and ∞. Then $h^{-1} \circ f \circ h$ fixes 0 and ∞. Moreover, for every isometry f' fixing the same two points as f, the isometry $h^{-1} \circ f' \circ h$ fixes 0 and ∞. This implies that for answering the question it is sufficient to consider isometries fixing 0 and ∞.

Now, any orientation-preserving isometry of the upper half-plane can be written as $\frac{az+b}{cz+d}$ with real a, b, c, d. Preserving 0 and ∞ means $b = 0$ and $c = 0$, so any orientation-preserving isometry fixing 0 and ∞ can be written as $z \mapsto az$, $a \in \mathbb{R}^+$. Hence, we get a one-parameter family of orientation-preserving isometries (all hyperbolic).

Similarly, we obtain a one-parameter family of orientation-reversing isometries $-a\bar{z}$, $a \in \mathbb{R}^+$.

14.5. ($\star$)

(a) Show that the group of isometries of the hyperbolic plane is generated by reflections.

(b) How many reflections do you need to map a triangle ABC to a congruent triangle $A'B'C'$?

Solution:

We can do (a) and (b) simultaneously, using the same procedure as in $\mathbb{E}^2$ or S^2: first apply reflection r_1 to take A to A' (with respect to the perpendicular bisector of AA'); then use the reflection r_2 in $A'M$ (where M is the midpoint of $B'r_1(B)$); finally, if needed, apply the reflection in $A'B'$. Thus, we need at most 3 reflections.

14.6. ($\star$)

(a) Does there exist a regular triangle on hyperbolic plane?

(b) Does there exist a right-angled regular polygon on hyperbolic plane? How many edges does it have (if exists)?

Solution:

(a) In the Poincaré disc model, consider a triangle whose vertices are represented by vertices of a regular Euclidean triangle with centre at O. By symmetry, this triangle is also a regular hyperbolic triangle (in other words, all Euclidean isometries we would use to check that the Euclidean triangle is regular are also isometries of the hyperbolic plane).

(b) The same construction as in (a) shows that there is a regular n-gon for all integer $n \geq 3$ in hyperbolic plane. When we make this n-gon very small, its sides are almost Euclidean lines, so its angles are almost the same as the angles of a regular Euclidean n-gon, i.e. $(n-2)\pi/n$. When the regular Euclidean n-gon grows, the angles of the corresponding hyperbolic n-gon decrease monotonically (to see this use Question 13.4), when all vertices of the n-gon are on the absolute, the angles are 0. So, the angles take every intermediate value between $(n-2)\pi/n$ and 0. In particular, if $n > 4$ then $(n-2)\pi/n > \pi/2$, which implies that there is a right-angled n-gon for every $n > 4$. We also know (from the sum of angles) that there are no right angled triangles and quadrilaterals.

14.7. (a) Show that the angle bisectors in a hyperbolic triangle are concurrent.

(b) Show that every hyperbolic triangle has an inscribed circle.

(c) Does every hyperbolic triangle have a circumscribed circle?

Solution:

(a) Similarly to Euclidean/spherical cases, an angle bisector is a locus of points that lie on the same distance from the rays forming the angle (this is most clear if you put the vertex of the angle at the centre of the Poincare disc model). So, the intersection point of two angle bisectors lies on the same distance from all three sides of the triangle, which implies that it actually lies on the third angle bisector.

(The intersection point does exists since the two ends of one angle bisector – say AA_1 – lie on two different sides of the other angle: $A \in BA$, $A_1 \in BC$, so they are separated by the angle bisector BB_1).

(b) Take the circle centred at the point of intersection of angle bisectors with radius equal to the distance to each of the three sides – this is the inscribed circle.

(c) Consider the Euclidean circle γ passing through the vertices of the triangle. If γ lies entirely inside the hyperbolic plane (i.e. in the disc or in the upper half-plane) then it represents some hyperbolic circle passing through the given points. However, the circle γ may intersect the boundary of hyperbolic plane. Then it does not represent any hyperbolic circle. Moreover, in the latter case no hyperbolic circle passes through the given points.