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Geometry III/IV, Solutions: weeks 15–16

Elementary hyperbolic geometry

15.1. (a) Let P and Q be the feet of the altitudes in an ideal hyperbolic triangle. Find PQ.

(b) Find the radius of a circle inscribed into an ideal hyperbolic triangle.

(c) Show that a radius of a circle inscribed into a hyperbolic triangle does not exceed $\operatorname{arccosh}(2/\sqrt{3})$.

Solution:

(a) We use the upper half-plane model. Let $X = -1$, $Y = 1$, $Z = \infty$ (we can assume that since all ideal triangles are congruent). By symmetry reasons, we can also assume that P and Q lie on XZ and YZ respectively. The line through Y orthogonal to XZ is represented by an arc of the circle $|z + 1| = 2$, so $P = -1 + 2i$. Similarly, $Q = 1 + 2i$. Hence,

$$\cosh d(P, Q) = 1 + \frac{4}{2 \cdot 2 \cdot 2} = \frac{3}{2}.$$

(b) The centre I of the inscribed circle is the intersection of three altitudes (this becomes clear if we place the ideal triangle in the Poincaré disc so that the vertices form a regular Euclidean triangle). One of the altitudes is the circle $|z + 1| = 2$, another is the line $\operatorname{Re} z = 0$. So, $I = i\sqrt{3}$. The required radius r is the distance from I to (any) foot of an altitude, say to $R = i$. Hence,

$$\cosh r = 1 + \frac{(\sqrt{3} - 1)^2}{2\sqrt{3}} = 1 + \frac{3 - 2\sqrt{3} + 1}{2\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

Alternatively, one can observe that the inscribed circle is represented by the Euclidean circle $|z-2i| = 1$, and the segment of the line $\operatorname{Re} z = 0$ between the two intersection points with the circle is a diameter. Thus, the radius is the half of the distance between points i and $3i$, i.e. $r = (\ln 3)/2$.

(c) We will show that any triangle ABC can be enclosed by some ideal triangle. Notice that the inscribed circle is the largest circle sitting inside a given triangle. So, the radius of the inscribed circle of $\triangle ABC$ does not exceed the radius of the inscribed circle of the ideal triangle (which is $(\ln 3)/2$, as computed in (b)).

Let $X, Y \in \partial H^2$ be the endpoints of the line AB, and let $Z \in \partial H^2$ be the second endpoint of the line XC. Then $\triangle ABC$ lies inside the ideal triangle XYZ.

15.2. For a right hyperbolic triangle ($\gamma = \frac{\pi}{2}$) show:

$$\text{(a)}\ \tanh b = \tanh c \cos \alpha, \qquad \text{(b)}\ \sinh a = \sinh c \sin \alpha.$$

Solution:

We will use the same notation as in the proof of Theorem 6.21 (Pythagorean Theorem), see Fig. 1. Also, we will use the values $\cosh b = \frac{1+k^2}{2k}$ and $\cosh c = \frac{1+k^2}{2k \sin \varphi}$ computed in the proof of Theorem 6.21.

First, we show

$$\sin^2 \alpha = \frac{4k^2 \cos^2 \varphi}{(k + 1)^2 - 4k^2 \sin^2 \varphi} = \frac{4k^2 \cos^2 \varphi}{(k^2 - 1)^2 + 4k^2 \cos^2 \varphi}. \qquad (1)$$

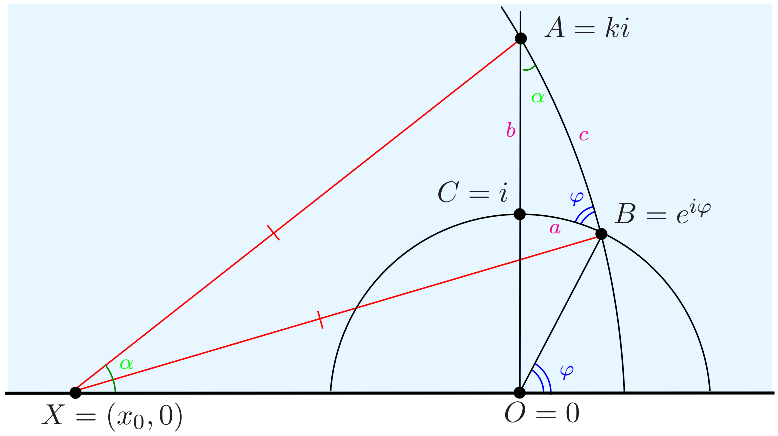


Figure 1: Notation for Problem 15.2

Let $X = (x_0, 0)$ be the (Euclidean) centre of the (Euclidean) circle representing the hyperbolic line AB. Then $\alpha = \angle AXO$ (as XA is a radius, so is orthogonal to the circle and the horizontal line XO is orthogonal to the vertical line AC). So,

$$\sin^2 \alpha = \sin^2 \angle AXO = \frac{k^2}{k^2 + x_0^2}.$$

To find x_0, notice that $XA = XB$ (as X is the centre of the circle), which implies

$$x_0^2 + k^2 = (\cos^2 \varphi - x_0)^2 + \sin^2 \varphi \qquad \Leftrightarrow \qquad k^2 = 1 - 2x_0 \cos \varphi,$$

i.e.

$$x_0 = \frac{1 - k^2}{2 \cos \varphi}.$$

Hence,

$$\sin^2 \alpha = \sin^2 \angle AXO = \frac{k^2}{k^2 + x_0^2} = \frac{k^2}{k^2 - (\frac{k^2-1}{2\cos\varphi})^2} = \frac{4k^2 \cos^2 \varphi}{(k^2 - 1)^2 + 4k^2 \cos^2 \varphi} =$$

$$\frac{4k^2 \cos^2 \varphi}{(k^2 + 1)^2 - 4k^2 + 4k^2 \cos^2 \varphi} = \frac{4k^2 \cos^2 \varphi}{(k^2 + 1)^2 - 4k^2 \sin^2 \varphi}$$

(a) Using the expressions for $\cosh b$ and $\cosh c$ we get respectively

$$\tanh^2 b = \frac{\sinh^2 b}{\cosh^2 b} = \frac{\cosh^2 b - 1}{\cosh^2 b} = 1 - \frac{1}{\cosh^2 b} = 1 - \frac{4k^2}{(1 + k^2)} = \left(\frac{1 - k^2}{1 + k^2}\right)^2$$

and

$$\tanh^2 c = 1 - \frac{1}{\cosh^2 c} = 1 - \frac{4k^2 \sin^2 \varphi}{(1 + k^2)^2} = \frac{(1 - k^2)^2 + 4k^2 \cos^2 \varphi}{(1 + k^2)^2}.$$

On the other hand,

$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \frac{4k^2 \cos^2 \varphi}{(k^2 - 1)^2 + 4k^2 \cos^2 \varphi} = \frac{(k^2 - 1)^2}{(k^2 - 1)^2 + 4k^2 \cos^2 \varphi},$$

which clearly satisfies the required identity $\tanh^2 b = \tanh^2 c \cos^2 \alpha$.

(b) Similarly, using the expressions for $\cosh a$ and $\cosh c$ we get respectively

$$\sinh^2 a = \cosh^2 a - 1 = \frac{1}{\sin^2\varphi} - 1 = \frac{\cos^2\varphi}{\sin^2\varphi}$$

and

$$\sinh^2 c = \cosh^2 c - 1 = (\frac{1+k^2}{2k\sin\varphi})^2 - 1 = \frac{(k^2+1)^2 - 4k^2\sin^2\varphi}{4k^2\sin^2\varphi}.$$

Hence, comparing to (1), we get $\sinh a = \sinh c \sin\alpha$.

15.3. Show that in the upper half-plane model the following distance formula holds:

$$4\sinh^2\frac{d}{2} = \frac{|z-w|^2}{\mathrm{Im}(z)\mathrm{Im}(w)}.$$

Solution:

$$\sinh^2\frac{d}{2} = (\frac{e^{d/2}+e^{-d/2}}{2})^2 = \frac{e^d + e^{-d} - 2}{4} = \frac{1}{2}(\cosh d - 1) = \frac{1}{2}\frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}.$$

15.4. Find an area of a right-angled hyperbolic pentagon.

Solution:

Subdividing the pentagon into 3 triangles, we see that $S = 3\pi - 5\frac{\pi}{2} = \frac{\pi}{2}$.

15.5. ($\star$) In the upper half-plane model, find the locus of points that lie on distance d from the line $\{\mathrm{Re}\, z = 0\}$.

Solution:

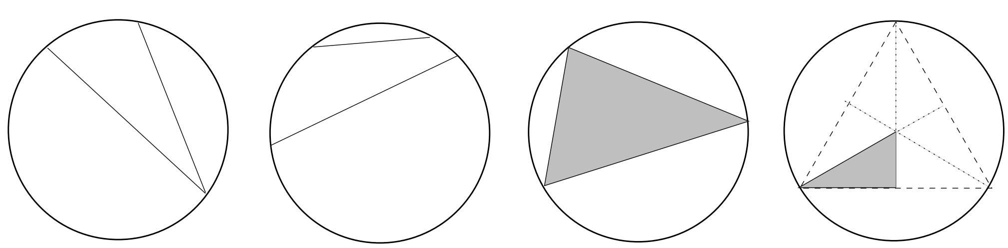
Consider the isometry $z \mapsto kz$ for $k > 0$. Let z_0 be a point on distance d from the line 0∞. Then every point kz_0 lies on the same distance from 0∞. So, we get a (Euclidean) ray lying in the locus. Now, applying reflection with respect to the imaginary axis $z \mapsto -\bar{z}$, we see that the locus contains also all points on another Euclidean ray $-k\bar{z}_0$.

Let us prove now that the locus contains no other points except the two rays described above. The distance from a point A to a line l is the length of the segment AH perpendicular to l, $H \in l$. Clearly, each line perpendicular to l contains exactly two points on the given distance d from l (one point in each half-plane). All lines perpendicular to 0∞ are represented by circles centred in 0, and each of them intersects each of the two rays. So, there are no other points in the locus.

Projective models

16.1. In the Klein disc model draw two parallel lines, two ultra-parallel lines, an ideal triangle, a triangle with angles $(0, \frac{\pi}{2}, \frac{\pi}{3})$.

Solution:



16.2. ($\star$) Show that three altitudes of a hyperbolic triangle either have a common point, or are all parallel to each other, or there exists a unique line orthogonal to all three altitudes.

Solution:

First, assume the triangle is ideal. Then we can place its vertices to vertices of an equilateral Euclidean triangle on the absolute (say, in the Klein disc), and thus the altitudes intersect at the origin by symmetry.

Therefore, we can now assume a vertex A belongs to $\mathbb{H}^2$, so without loss of generality we may assume that A is the centre of the Klein disc and B and C are any two other points in $\overline{\mathbb{H}}^2$. Let AH_a, BH_b and CH_c be the (Euclidean) altitudes of the Euclidean triangle with vertices A, B, C. Then AH_a, BH_b and CH_c are also (hyperbolic) altitudes of hyperbolic triangle ABC. Indeed, $AH_a \perp BC$ since AH_a lie on a diameter of the disc, $BH_b \perp AC$ and $CH_c \perp AB$ since AC and AB lie on the diameter of the disc.

Being altitudes of a Euclidean triangle, the lines AH_a, BH_b and CH_c have a common point T, however, T doesn't necessarily belongs to the disc. If T lies in the disc, the altitudes of $\triangle ABC$ have a common point. If T lies on the boundary of the disc, then the altitudes of $\triangle ABC$ are all parallel. Finally, if T lies outside the disc, then there exists a unique (hyperbolic) line l orthogonal to all three altitudes (to find this line l consider the (Euclidean) lines t_1 and t_2 passing through T and tangent to the boundary of the disc, then l is the line through the points $t_1 \cap \partial\mathbb{H}^2$ and $t_2 \cap \partial\mathbb{H}^2$).

16.3. Let $\boldsymbol{u}, \boldsymbol{v}$ be two vectors in $\mathbb{R}^{2,1}$. Denote $Q = |\frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}|$, where $(x, y) = x_1y_1 + x_2y_2 - x_3y_3$. Show the following distance formulae:

(a) if $(\boldsymbol{u}, \boldsymbol{u}) < 0$, $(\boldsymbol{v}, \boldsymbol{v}) < 0$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ define two points in $\mathbb{H}^2$, and $Q = \cosh^2 d(\boldsymbol{u}, \boldsymbol{v})$.

(b) if $(\boldsymbol{u}, \boldsymbol{u}) < 0$, $(\boldsymbol{v}, \boldsymbol{v}) > 0$, then $\boldsymbol{u}$ defines a point and $\boldsymbol{v}$ defines a line $l_{\boldsymbol{v}}$ in $\mathbb{H}^2$, and $Q = \sinh^2 d(\boldsymbol{u}, l_{\boldsymbol{v}})$.

(c) if $(\boldsymbol{u}, \boldsymbol{u}) > 0$, $(\boldsymbol{v}, \boldsymbol{v}) > 0$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ define two lines $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ in $\mathbb{H}^2$ and
- if $Q < 1$, then $l_{\boldsymbol{u}}$ intersects $l_{\boldsymbol{v}}$ forming angle φ satisfying $Q = \cos^2 \varphi$;
- if $Q = 1$, then $l_{\boldsymbol{u}}$ is parallel to $l_{\boldsymbol{v}}$;
- if $Q > 1$, then $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ are ultra-parallel lines satisfying $Q = \cosh^2 d(l_{\boldsymbol{u}}, l_{\boldsymbol{v}})$.

Solution:

We will compute in the hyperboloid model. Moreover, we will use the isometry group to reduce the problem to a 2-dimensional one.

(a) By transitivity of the isometry group on $\mathbb{H}^2$ we may assume that $\boldsymbol{u} = (0, 0, 1)$. Applying a rotation around this point (in the 3-dimensional space it is represented by a rotation around the third coordinate axis) we may assume that $\boldsymbol{v} = (v_1, 0, v_3)$, $v_1^2 - v_3^2 = -1$. We will also assume $v_1 > 0$.
We find $d(\boldsymbol{u}, \boldsymbol{v})$ by definition, as a cross-ratio of four lines.
The line (plane in the model) through $\boldsymbol{u}$ and $\boldsymbol{v}$ has the equation $x_2 = 0$, i.e. it is the line $(\boldsymbol{x}, \boldsymbol{a}) = 0$ for the vector $\boldsymbol{a} = (0, 1, 0)$. This line intersects the absolute at the points $(\boldsymbol{x}, \boldsymbol{x}) = 0$, $x_2 = 0$, i.e. in $x_1^2 - x_3^2 = 0$ which gives two solutions for $x_3 > 0$: $\boldsymbol{X} = (-1, 0, 1)$ and $\boldsymbol{Y} = (1, 0, 1)$.To find the distance $d(\boldsymbol{u}, \boldsymbol{v})$ we need to find a cross-ratio of four lines spanned by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{X}$ and $\boldsymbol{Y}$.
To find the cross-ratio of four lines we intersect all four lines by some line l (the result does not depend on the choice of l). Choose l to be the horizontal line through $(0, 0, 1)$ (it is given by equations $x_3 = 1$, $x_2 = 0$). Renormalising $\boldsymbol{v} = (v_1, 0, v_3)$ so that it belongs to the plane $x_3 = 1$ we get $\boldsymbol{v}' = (\frac{v_1}{v_3}, 0, 1)$. So, using the line $x_3 = 1$, $x_2 = 0$ we get

$$\big|[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{Y}, \boldsymbol{X}]\big| = \big|[0, \frac{v_1}{v_3}, 1, -1]\big| = \Big|\frac{1-0}{1-\frac{v_1}{v_3}} \Big/ \frac{-1-0}{-1-\frac{v_1}{v_3}}\Big| = \Big|\frac{v_1+v_3}{v_1-v_3}\Big| = \Big|\frac{(v_1+v_3)^2}{v_1^2-v_3^2}\Big| = (v_1+v_3)^2,$$

so that

$$d(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}\big|\ln|[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{X}, \boldsymbol{Y}]|\big| = \frac{1}{2}\ln(v_1+v_3)^2 = \ln(v_1+v_3),$$

which implies $e^d = v_1 + v_3$, and

$$\cosh d = \frac{v_1 + v_3 + \frac{1}{v_1+v_3}}{2} = \frac{v_3 + \frac{1+v_1(v_1+v_3)}{v_1+v_3}}{2} = \frac{v_3 + \frac{1+v_1^2+v_1v_3)}{v_1+v_3}}{2} = \frac{v_3 + \frac{v_3^2+v_1v_3)}{v_1+v_3}}{2} = v_3.$$

On the other hand,

$$\frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})} = \frac{v_3^2}{(-1)(-1)} = v_3^2.$$

Thus,

$$\cosh^2 d(\boldsymbol{u},\boldsymbol{v}) = \left|\frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}\right|.$$

(b) Let $\boldsymbol{t} \in l_{\boldsymbol{v}}$ be an orthogonal projection of $\boldsymbol{u}$ to $l_{\boldsymbol{v}}$, i.e. the line $\boldsymbol{tu}$ is perpendicular to $l_{\boldsymbol{v}}$. Clearly, $d(\boldsymbol{u}, l_{\boldsymbol{v}}) = d(\boldsymbol{u},\boldsymbol{t})$.
Without loss of generality we may assume that $\boldsymbol{u} = (0,0,1)$ and $\boldsymbol{t} = (t_1, 0, t_3)$, $t_1^2 - t_3^2 = -1$. By part (a),

$$\cosh^2 d(\boldsymbol{u}, l_{\boldsymbol{v}}) = \cosh^2 d(\boldsymbol{u},\boldsymbol{t}) = \left|\frac{t_3^2}{(-1)(-1)}\right| = t_3^2.$$

Therefore,

$$\sinh^2 d(\boldsymbol{u}, l_{\boldsymbol{v}}) = \cosh^2 d(\boldsymbol{u}, l_{\boldsymbol{v}}) - 1 = t_3^2 - 1 = t_1^2.$$

Now, let us find the equation for the line $l_{\boldsymbol{v}}$. The line $\boldsymbol{tu}$ corresponds to the plane given by the equation $x_2 = 0$. The whole pattern (i.e. hyperboloid, the point $\boldsymbol{u}$, the line $l_{\boldsymbol{v}}$ the line $\boldsymbol{tu}$) is symmetric with respect to this plane. Hence, the vector $\boldsymbol{v}$ defining the line $l_{\boldsymbol{v}}$ has zero second coordinate $v_2 = 0$, which implies $\boldsymbol{v} = (v_1, 0, v_3)$. Since the line $l_{\boldsymbol{v}}$ contains the point $\boldsymbol{t} = (t_1, 0, t_3)$, we have $(\boldsymbol{v},\boldsymbol{t}) = 0$, i.e. $v_1 t_1 - v_3 t_3 = 0$. This implies $\boldsymbol{v} = \lambda(t_3, 0, t_1)$, or simply $\boldsymbol{v} = (t_3, 0, t_1)$ after rescaling $(\boldsymbol{v},\boldsymbol{v}) = 1$. Hence,

$$\left|\frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}\right| = \left|\frac{t_1^2}{(-1)\cdot 1}\right| = t_1^2,$$

which coincides with the value of $\sinh^2 d(\boldsymbol{u}, l_{\boldsymbol{v}})$.

(c)
- $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ have a common point in $\mathbb{H}^2$.
  Applying an isometry, we may assume that the point of intersection of $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ is $(0,0,1)$. Then the planes through the origin representing the lines $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ are vertical planes (passing through the third coordinate axis), these planes are represented by vectors $(u_1, u_2, 0)$, $(v_1, v_2, 0)$ (to see that, notice that the vertical planes are symmetric with respect to the plane $x_3 = 0$). Furthermore, due to the rotational symmetry, the angles at the point $(0,0,1)$ are Euclidean angles, i.e. φ (or $\pi - \varphi$) coincides with the angle between $(u_1, u_2, 0)$ and $(v_1, v_2, 0)$. By Euclidean formula for computation of angles we get

  $$\cos\varphi = \pm\frac{(\boldsymbol{u},\boldsymbol{v})}{\sqrt{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}}$$

  (we may use pseudo-scalar product $(\cdot,\cdot)$ in a Euclidean formula since the third coordinate is zero).
- $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ are ultra-parallel.
  Let h be a line orthogonal to both $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$. Let $h_{\boldsymbol{u}} = h \cap l_{\boldsymbol{u}}$ and $h_{\boldsymbol{v}} = h \cap l_{\boldsymbol{v}}$ be the intersection points. Then $d(l_{\boldsymbol{u}}, l_{\boldsymbol{v}}) = d(h_{\boldsymbol{u}}, h_{\boldsymbol{v}})$.
  Without loss of generality we may assume $h_{\boldsymbol{u}} = (0,0,1)$ and $h_{\boldsymbol{v}} = (t_1, 0, t_3)$, $t_1^2 - t_3^2 = 1$ (so that h corresponds to the plane $x_2 = 0$). Then $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ are represented by the vectors $\boldsymbol{u} = (1,0,0)$ and $\boldsymbol{v} = (t_3, 0, t_1)$ (since $(h_{\boldsymbol{v}}, \boldsymbol{v}) = 0$ and $v_2 = 0$). This implies that

  $$\cosh^2 d(h_{\boldsymbol{u}}, h_{\boldsymbol{v}}) = \left|\frac{(h_{\boldsymbol{u}}, h_{\boldsymbol{v}})^2}{(h_{\boldsymbol{u}}, h_{\boldsymbol{u}})(h_{\boldsymbol{v}}, h_{\boldsymbol{v}})}\right| = \frac{t_3}{|t_1^2 - t_3^2|} = \left|\frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}\right|,$$

  This proves the statement since $d(l_{\boldsymbol{u}}, l_{\boldsymbol{v}}) = d(h_{\boldsymbol{u}}, h_{\boldsymbol{v}})$.
- $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ are parallel.
  The result for this case follows from two previous ones by continuity.
  Alternatively, one can use an isometry to place the common point of two lines at $(1,0,1)$, and the other endpoints of $l_{\boldsymbol{u}}$ and $l_{\boldsymbol{v}}$ at $(-1,0,1)$ and $(0,1,1)$ respectively, i.e. $\boldsymbol{u} = (0,1,0)$ and $\boldsymbol{v} = (1,1,1)$. Then

  $$Q = \left|\frac{(\boldsymbol{u},\boldsymbol{v})^2}{(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v})}\right| = \frac{1}{1\cdot 1} = 1.$$

16.4. ($\star$) Consider the two-sheet hyperboloid model $\{\boldsymbol{u} = (u_1, u_2, u_3) \in \mathbb{R}^{2,1} \mid (\boldsymbol{u}, \boldsymbol{u}) = -1,\ u_3 > 0\}$, where $(\boldsymbol{u}, \boldsymbol{u}) = u_1^2 + u_2^2 - u_3^2$.

(a) For the vectors

$$\begin{array}{lll} \boldsymbol{v}_1 = (2,1,2) & \boldsymbol{v}_2 = (0,1,2) & \boldsymbol{v}_3 = (3,4,5) \\ \boldsymbol{v}_4 = (1,0,0) & \boldsymbol{v}_5 = (0,1,0) & \boldsymbol{v}_6 = (1,1,2) \end{array}$$

decide whether $\boldsymbol{v}_i$ defines a point in $\mathbb{H}^2$, a point on the absolute, or a line in $\mathbb{H}^2$.

(b) Find the distance between the two points of $\mathbb{H}^2$ described in (a).

(c) Which pairs of lines in (a) are intersecting? Which lines are parallel? Which lines are ultra-parallel? Justify your answer.

(d) Find the distances between all pairs of ultra-parallel lines in (a).

(e) Does any of the points in (a) lie on any of the lines above?

(f) Find the angles between the pairs of intersecting lines.

Solution:

(a) We need to check (v_i, v_i): if it is negative, v_i corresponds to a point of hyperbolic plane, if it is equal to zero, v_i is a point of the absolute, if it is positive, then v_i corresponds to a line (more precisely, it is a normal vector to plane through $(0,0,0)$ which determines a line in the model).

$$\begin{array}{ll} (v_1, v_1) = 4 + 1 - 4 = 1 > 0, & \text{line;} \\ (v_2, v_2) = 0 + 1 - 4 = -3 < 0, & \text{point} \\ (v_3, v_3) = 9 + 16 - 25 = 0, & \text{point of the absolute;} \\ (v_4, v_4) = 1 + 0 - 0 = 1 > 0, & \text{line;} \\ (v_5, v_5) = 0 + 1 - 0 = 1 > 0, & \text{line;} \\ (v_6, v_6) = 1 + 1 - 4 = -2 < 0, & \text{point.} \end{array}$$

(b)

$$\cosh^2(d(v_2, v_6)) = \frac{(v_2, v_6)^2}{(v_2, v_2)(v_6, v_6)} = \frac{(0+1-4)^2}{(-3)(-2)} = \frac{9}{6} = \frac{3}{2}.$$

So, $d(v_2, v_6) = \operatorname{arccosh} \sqrt{\frac{3}{2}}$.

(c) $|\frac{(v_1,v_4)^2}{(v_1,v_1)(v_4,v_4)}| = \frac{4}{1\cdot 1} = 4 > 1$, so, v_1 and v_4 are ultra-parallel lines.
$|\frac{(v_1,v_5)^2}{(v_1,v_1)(v_5,v_5)}| = \frac{1}{1\cdot 1} = 1$, so, v_1 is parallel to v_5.
$|\frac{(v_4,v_5)^2}{(v_4,v_4)(v_5,v_5)}| = \frac{0}{1\cdot 1} = 0 < 1$, so, v_4 intersects v_5.

(d) $\cosh^2(d) = ||\frac{(v_1,v_4)^2}{(v_1,v_1)(v_4,v_4)}| = 4$, so, $d = \operatorname{arccosh} 2$.

(e) A point v_i lies on a line v_j if and only if $(v_i, v_j) = 0$.
This holds for the point v_2 and the line v_4.
This also holds for the point of the absolute v_3 and the line v_1.

(f) $\cos^2 \alpha = |\frac{(v_4,v_5)^2}{(v_4,v_4)(v_5,v_5)}| = 0$, so, the lines are orthogonal.

References:

Lectures (Elementary hyperbolic geometry, area, Klein model and hyperboloid model) are based on Lectures VII, VIII, VI and XIII of Prasolov's book.