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Geometry III/IV, Solutions: weeks 17–18

Isometries of the hyperbolic plane

17.1. Show that any pair of parallel lines can be transformed to any other pair of parallel lines by an isometry.

Solution:

Consider a pair of parallel lines l_1 and l_2. Let X be the common point (lying at the absolute) of these lines and Y_1 and Y_2 be other endpoints of these lines. By triple transitivity of the action of isometries on points of the absolute, we can see that X, Y_1, Y_2 can be taken to the endpoints of any other pair of parallel lines.

Remark: another option is just to look at these line in the upper half-plane, assuming $X = \infty$, and to construct the required isometry explicitly.

17.2. Let $A, B \in \gamma$ be two points on a horocycle γ. Show that the perpendicular bisector to the segment AB (of a hyperbolic line!) is orthogonal to γ.

Solution:

Consider the upper half-plane model, let ∞ be the centre of the horocycle. Then γ is represented by a horizontal (Euclidean) line, the perpendicular bisector to AB is represented by a vertical ray, which is obviously orthogonal to the horocycle.

17.3. Let f be a composition of three reflections. Show that f is a glide reflection, i.e. a hyperbolic translation along some line composed with a reflection with respect to the same line.

Solution:

Consider first the restriction of f to the absolute (parametrized by the angle $\varphi \in [0, 2\pi)$). As f is orientation-reversing, the function $f(\varphi)$ (considered modulo 2π) is monotonically decreasing. Hence, there are exactly two points where $f(\varphi) = \varphi$ (the intersection points of the graph of f with the diagonal). This implies that f preserves two points of the absolute.

Now, in Problem 14.4 we have already classified all isometries preserving two points of the absolute. In particular, for the orientation-reversing case we have seen that there is a one-parameter family of such isometries, and that in the upper half-plane (with 0 and ∞ fixed) any such isometry can be written as $z \mapsto -a\bar{z}$, $a \in R_+$. Notice that this is a composition of a hyperbolic translation along the line $\{\mathrm{Re}\, z = 0\}$ and the reflection with respect to the same line.

17.4. ($\star$) Let f be an isometry of the hyperbolic plane such that the distance from A to $f(A)$ is the same for all points $A \in \mathbb{H}^2$. Show that f is an identity map.

Solution:

This follows from the classification of isometries. If f is not an identity, then it is of one of five types: elliptic, parabolic, hyperbolic, reflection or glide reflection.

Elliptic isometries and reflections have fixed points in $\mathbb{H}^2$ (i.e., there are points such that the distance from A to $f(A)$ is zero), so this is not the case.

Hyperbolic isometries and glide reflections preserve equidistant curves, and it is easy to see (say, in the upper half-plane model conjugating f to either $z \mapsto kz$ or $z \mapsto -k\bar{z}$) that the distance between $f(A)$ and A is an increasing function on the distance from A to the axis of f.

Finally, parabolic isometries preserve horocycles, and, again, considering f in the upper half-plane model (and conjugating it to $z \mapsto z+1$) it is easy to see that the distance between $f(A)$ and A is an increasing function on the "distance" from A to the fixed point of f (although the distance is infinite and thus is not defined, one can *compare* distances from different points to a point on the absolute by considering appropriate horocycles).

17.5. ($\star$) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors in the hyperboloid model such that $(\boldsymbol{a},\boldsymbol{a})>0$ and $(\boldsymbol{b},\boldsymbol{b})>0$. Let $l_{\boldsymbol{a}}$ and $l_{\boldsymbol{b}}$ be the lines determined by equations $(\boldsymbol{x},\boldsymbol{a})=0$ and $(\boldsymbol{x},\boldsymbol{b})=0$ respectively, and let $r_{\boldsymbol{a}}$ and $r_{\boldsymbol{b}}$ be the reflections with respect to $l_{\boldsymbol{a}}$ and $l_{\boldsymbol{b}}$.

(a) For $\boldsymbol{a}=(0,1,0)$ and $\boldsymbol{b}=(1,0,0)$ write down $r_{\boldsymbol{a}}$ and $r_{\boldsymbol{b}}$. Find $r_{\boldsymbol{b}}\circ r_{\boldsymbol{a}}(\boldsymbol{v})$, where $\boldsymbol{v}=(0,1,2)$.

(b) What is the type of the isometry $\varphi=r_{\boldsymbol{b}}\circ r_{\boldsymbol{a}}$ for $\boldsymbol{a}=(1,1,1)$ and $\boldsymbol{b}=(1,1,-1)$? (Hint: you don't need to compute $r_{\boldsymbol{a}}$ and $r_{\boldsymbol{b}}$).

(c) Find an example of $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\varphi=r_{\boldsymbol{b}}\circ r_{\boldsymbol{a}}$ is a rotation by $\pi/2$.

Solution:

(a) $r_a(\boldsymbol{x})=\boldsymbol{x}-2\frac{(\boldsymbol{x},\boldsymbol{a})}{(\boldsymbol{a},\boldsymbol{a})}\boldsymbol{a}=\boldsymbol{x}-2x_2\boldsymbol{a}, \quad r_{\boldsymbol{b}}(\boldsymbol{x})=\boldsymbol{x}-2\frac{(\boldsymbol{x},\boldsymbol{b})}{(\boldsymbol{b},\boldsymbol{b})}\boldsymbol{b}=\boldsymbol{x}-2x_1\boldsymbol{b};$
$(\boldsymbol{a},\boldsymbol{a})=1$, $(\boldsymbol{v},\boldsymbol{a})=1$, so,
$\boldsymbol{u}:=r_{\boldsymbol{a}}(\boldsymbol{v})=r_{\boldsymbol{a}}((0,1,2))=(0,1,2)-2\frac{1}{1}(0,1,0)=(0,-1,2).$
$(\boldsymbol{b},\boldsymbol{b})=1$, $(\boldsymbol{u},\boldsymbol{b})=0$, so,
$r_{\boldsymbol{b}}\circ r_{\boldsymbol{a}}(\boldsymbol{v})=r_{\boldsymbol{b}}(\boldsymbol{u})=(0,-1,2)-0=(0,-1,2).$

(b) To find the type of isometry $\varphi=r_{\boldsymbol{b}}\circ r_{\boldsymbol{a}}$ it is sufficient to determine whether the lines $l_{\boldsymbol{a}}$ and $l_{\boldsymbol{b}}$ are intersecting, or parallel, or ultraparallel:

- if they do intersect $\quad\varphi$ is elliptic;
- if they are parallel $\quad\varphi$ is parabolic;
- if they are ultraparallel $\quad\varphi$ is hyperbolic.

The behavior of two lines is determined by the value $Q=\frac{(\boldsymbol{a},\boldsymbol{b})^2}{(\boldsymbol{a},\boldsymbol{a})(\boldsymbol{b},\boldsymbol{b})}$:

- $l_{\boldsymbol{a}}$ intersects $l_{\boldsymbol{b}}$ if $Q<1$;
- $l_{\boldsymbol{a}}$ is parallel to $l_{\boldsymbol{b}}$ if $Q=1$;
- $l_{\boldsymbol{a}}$ is ultraparallel to $l_{\boldsymbol{b}}$ if $Q>1$.

In our case, $Q=\frac{9}{1\cdot 1}>1$, so that the lines are ultraparallel. This implies that φ is hyperbolic.

(c) To get a rotation by $\pi/2$ we need to find two lines making the angle $\pi/4$. The easiest way to get such a pair of lines is to put their intersection into the centre of the model where the angles do coincide with Euclidean ones.

Take the lines defined by $\boldsymbol{a}=(1,0,0)$ and $\boldsymbol{b}=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0))$. Then $\cos^2(\angle\boldsymbol{ab})=Q=\frac{(\frac{\sqrt{2}}{2})^2}{1\cdot 1}=\frac{2}{4}$. So, $\angle\boldsymbol{ab}=\arccos\frac{\sqrt{2}}{2}=\pi/4$.

Equidistant curves

18.1. Let l be a hyperbolic line, and let E_l be an equidistant curve for l.

(a) Let C_1 and C_2 be two connected components of the same equidistant curve E_l. Show that C_1 is also equidistant from C_2, i.e. given a point $A\in C_1$ the distance $d(A,C_2)$ from A to C_2 does not depend on the choice of A.

(b) Let $A \in E_l$ be a point on the equidistant curve, and let $A_l \in l$ be the point of l closest to A. Show that the line AA_l is orthogonal to the equidistant curve E_l.

(c) Let $P, Q \in l$ be two points on l, and let $A \in E_l$. Continue the rays AP and AQ till the next intersection points with E_l, denote the resulting intersection points by B and C. Let T be a curvilinear triangle ABC (with geodesic sides AB and AC, but BC being a segment of the equidistant curve). Assuming that all angles of ABC are acute, show that the area of T does not depend on the choice of $A \in E_l$.

(d) In the assumptions of (c), show that the area of the geodesic triangle ABC does not depend on the choice of A.

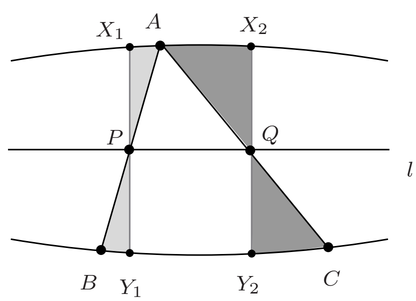
Solution:

(a) Any hyperbolic translation along the line l preserves both C_1 and C_2 (not pointwise) and moves A along C_1. Moreover, for any $B \in C_1$ there is a suitable translation T along l such that $T(A) = B$. So, the distance from B to C_2 is the same as $d(A, C_2)$.

(b) In the upper half-plane model, let l be a vertical ray on the line $x = 0$. Then the equidistant curve is the union of two rays from the origin, the line AA_l is represented by a half of a circle centred at the origin and is obviously orthogonal to the rays forming the equidistant curve. As the upper half-plane model is conformal, this implies that AA_l is orthogonal to E_l.

(c) Let l_P be the line through P orthogonal to l and let X_1 and Y_1 be the intersections of l_P with C_1 and C_2 respectively lying on distance c_0 from P. Similarly, we construct the line l_Q through Q, $l_Q \perp l$, and its intersection points X_2 and Y_2 with C_1 and C_2.

Consider the curvilinear triangles PAX_1 and PBY_1. The rotation R by π around P swaps these triangles (indeed, R preserves all lines through P and swaps the circles C_1 and C_2). This implies that these curvilinear triangles have equal areas. Similarly, the curvilinear triangles QAX_2 and QCY_2 have equal areas. So, the area of the curvilinear triangle ABC coincides with the area of curvilinear quadrilateral $X_1X_2Y_2Y_1$ (with geodesic sides X_1X_2 and Y_1Y_2, but sides X_1X_2 and Y_1Y_2 being the segments of the equidistant curve). The latter area does not depend on the choice of A. Notice, that here we use that ABC is acute-angled (if angle B or C is obtuse the diagram is more complicated).



(d) It is sufficient to prove that the distance between B and C does not depend on the choice of A (then the area of $\triangle ABC$ differs from the area of the curvilinear triangle ABC by the area of a lune BC formed by the geodesic segment and a segment of the equidistant curve).

To see that $d(B, C)$ is independent of the choice of A, consider the orthogonal projections A_l, B_l and C_l of the points A, B, C to the line l. Clearly, $d(B_l, P) = d(A_l, P)$ and $d(C_l, Q) = d(A_l, Q)$. This implies that $d(B_l, C_l) = 2d(P, Q)$, (here we use that ABC is acute-angled and hence, $A_l \in PQ$), which does not depend on A. Therefore, $d(B, C)$ does not depend on A.

18.2. ($\star$)

(a) Let l and l' be ultra-parallel lines. Let E_l be an equidistant curve for l intersecting l' in two points A and B. Denote by h the common perpendicular to l and l' and let $H = h \cap l'$ be the intersection point. Show that $AH = HB$.

(b) Let l be a line and E_l be an equidistant curve for l. For two points A, B on E_l, show that the perpendicular bisector of AB is also orthogonal to l.

(c) Let ABC be a triangle in the Poincaré disc model. Let γ be a Euclidean circumscribed circle (i.e. a circumscribed circle for ABC considered as a Euclidean triangle). Suppose that γ intersects the absolute at points X and Y. Show that the (hyperbolic) perpendicular bisector to AB is orthogonal to the hyperbolic line XY.

(d) Show that three perpendicular bisectors in a hyperbolic triangle are either concurrent, or parallel, of have a common perpendicular.

Solution:

(a) Consider the reflection r_h with respect to h. It preserves l (as $l \perp h$), and thus preserves E_l. By the same reason, r_h preserves l'. Thus, the intersection $A \in l' \cap E_l$ should be mapped by r_h to another point in $l' \cap E_l$, which is B. This implies that h is the perpendicular bisector of AB.

(b) This is just another wording of part (a). Let l' be the line AB, then we have proved that the common perpendicular to l and l' coincides with the perpendicular bisector of AB. In particular, the latter is orthogonal to l.

(c) The part of the curve γ lying inside $\mathbb{H}^2$ is an equidistant curve to the line XY (as it is a part of a Euclidean circle passing through the endpoints of XY). Therefore, it is orthogonal to XY by part (b).

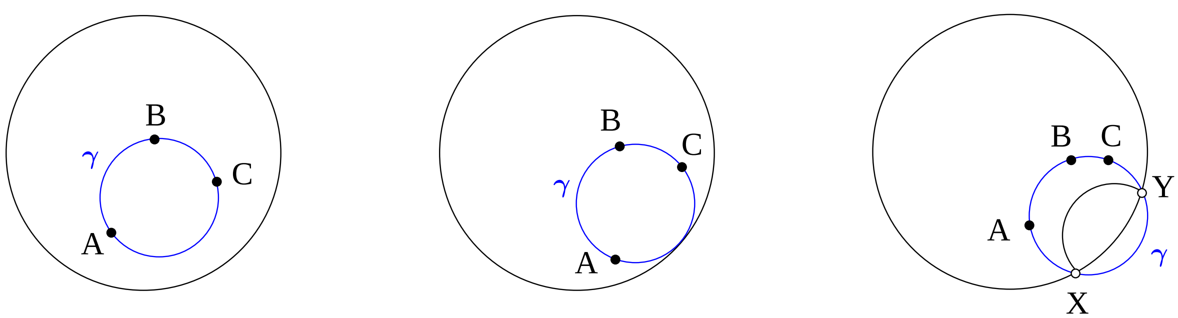
(d) Consider the triangle ABC in the Poincaré disc model. Let γ be the Euclidean circle through A, B, C. Consider three cases: γ lies inside hyperbolic plane, is tangent to the absolute or intersects the absolute at two different points.

If γ intersects the absolute at two points X and Y, then as shown in part (c) all perpendicular bisectors are orthogonal to XY.

If γ is tangent to the absolute at X, then γ is a horocycle centred at X. It is shown in Problem 17.2 that all perpendicular bisectors are orthogonal to γ, which implies they are passing through the centre X of γ (cf. the upper half-plane with $X = \infty$), and thus are parallel to each other.

If γ lies entirely inside the hyperbolic plane, it actually represents a hyperbolic circle. So, ABC has a circumscribed circle, whose centre is the point of concurrence of all three perpendicular bisectors.

Here are the diagrams showing what can happen in (c) and (d):



or, even more precisely: