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Epiphany 2020

Riemannian Geometry IV, Problems class 4 (Week 20): Solutions

P4.1. Let $c : [0, a] \to M$ be a geodesic. Find explicitly geodesic variations $F(s, t)$ and $F^0(s, t)$ of c such that their variational vector fields are non-orthogonal Jacobi fields $tc'(t)$ and $c'(t)$ respectively.

Solution:

Let $J(t) = tc'(t)$. Since $J(0) = 0$, we can apply the procedure used in the proof of Lemma 9.9. Namely, $F(s, t) = \exp_{c(0)} tv(s)$, where $v(s)$ is a curve in $T_{c(0)}M$ satisfying $v(0) = c'(0)$ and $v'(0) = \frac{D}{dt}J(0)$.

Observe that

$$\frac{D}{dt}J(0) = \frac{D}{dt}\bigg|_{t=0} tc'(t) = c'(0) + t\,\frac{D}{dt}\bigg|_{t=0} c'(t) = c'(0),$$

so the curve $v(s)$ should satisfy $v(0) = v'(0) = c'(0)$. We can take, for example, $v(s) = (s + 1)c'(0)$. Therefore, $F(s, t) = \exp_{c(0)}(tc'(0)(s + 1))$.

Now let $J(t) = c'(t)$. This Jacobi field does not vanish at $t = 0$, so we need to apply the method used in the proof of HW 6.2. Namely, $F^0(s, t) = \exp_{\gamma(s)} tV(s)$, where $\gamma(s)$ is a curve in M, $V(s)$ is a vector field along the curve $\gamma(s)$, and $\gamma(0) = c(0)$, $\gamma'(0) = J(0)$, $V(0) = c'(0)$, $\frac{D}{ds}V(0) = \frac{D}{dt}J(0)$.

For our particular $J(t)$ this implies that $\gamma(0) = c(0)$, $\gamma'(0) = c'(0)$, $V(0) = c'(0)$, $\frac{D}{ds}V(0) = 0$. Thus, we can take $\gamma(s) = c(s)$, $V(s) = c'(s)f(s)$, where $f(0) = 1$, $f'(0) = 0$. For example, $f(s) = 1 + s^2$ works. Therefore, we can take $F^0(s, t) = \exp_{c(s)}(tc'(s)(1 + s^2))$.

Note that both variations $F(s, t)$ and $F^0(s, t)$ lie entirely in the trace of c (which should not be surprising).

P4.2. Let $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ be the upper half-space model of the 3-dimensional hyperbolic space, where the metric is given by $(g_{ij}) = 1/z^2 I$. Given $a \in \mathbb{R}_{>0}$, show that the transformation $f_a : (x, y, z) \mapsto (ax, ay, az)$ is an isometry of $\mathbb{H}^3$.

Solution:

Let $\gamma(t)$ be a curve in $\mathbb{H}^3$, let us find $Df_{\gamma(0)}\gamma'(0)$:

$$Df_{\gamma(0)}\gamma'(0) = \frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) = \frac{d}{dt}\bigg|_{t=0} a\gamma(t) = a\gamma'(0),$$

which implies that $Df_{\gamma(0)} = aI$ (note that this doesn't depend on the point $\gamma(0)$). Therefore, for every $p = (x, y, z) \in \mathbb{H}^3$ and for every $v, w \in T_p\mathbb{H}^3$ we have

$$\langle Df(v), Df(w)\rangle_{f(p)} = \langle av, aw\rangle_{f(p)} = \frac{a^2\langle v, w\rangle_{\text{Eucl}}}{(az)^2} = \frac{\langle v, w\rangle_{\text{Eucl}}}{z^2} = \langle v, w\rangle_p.$$

P4.3. Show that the cone $z^2 = x^2 + y^2$ in $\mathbb{H}^3$ is isometric to Euclidean cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

Solution:

Let us first check that the sectional curvature on the cone M vanishes identically (formally speaking, this is not required, but at least indicates that the question makes sense).

Parametrize M by $(x, y, z) = (r\cos\varphi, r\sin\varphi, r)$, then

$$\begin{aligned} \frac{\partial}{\partial r} &= (\cos\varphi, \sin\varphi, 1) \\ \frac{\partial}{\partial \varphi} &= (-\sin\varphi, \cos\varphi, 0), \end{aligned}$$

so the metric is given by the matrix $(g_{ij}) = \frac{1}{r^2}\begin{pmatrix} 2 & 0 \\ 0 & r^2 \end{pmatrix} = \begin{pmatrix} 2/r^2 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, the only non-zero Christoffel symbol is $\Gamma^1_{11} = -1/r$, and thus the Riemann curvature tensor vanishes identically.

We now need to construct an isometry $f : M \to C$ between M and the cylinder C. Observe that any "horizontal slice" of M at height z is a circle of (Euclidean) radius z, and thus its length is equal to $(2\pi z)/z = 2\pi$. So, it would be natural if f mapped these slices to parallels of C. A natural candidate for the preimage of a meridian of C would be a generating curve of M parametrized by arc length.

Parametrize a generating curve of M by $c(x) = (\alpha(x), 0, \alpha(x))$. Then $c'(x) = (\alpha'(x), 0, \alpha'(x))$, so $\|c'(x)\|^2 = 2\alpha'(x)^2/\alpha(x)^2$. The quation $\|c'(x)\| = 1$ is then equivallent to $\alpha'(x) = \alpha(x)/\sqrt{2}$, so we can take $\alpha(x) = e^{x/\sqrt{2}}$.

Thus, we have a parametrization of M by $(e^{r/\sqrt{2}}\cos\varphi, e^{r/\sqrt{2}}\sin\varphi, e^{r/\sqrt{2}}) = e^{r/\sqrt{2}}(\cos\varphi, \sin\varphi, 1)$. The cylinder C is parametrized by $(\cos\vartheta, \sin\vartheta, z)$. Define a map $f : M \to C$ by $f(r, \varphi) = (z(r, \varphi), \vartheta(r, \varphi) = (r, \varphi)$, i.e.

$$f : e^{r/\sqrt{2}}(\cos\varphi, \sin\varphi, 1) \mapsto (\cos\varphi, \sin\varphi, r)$$

. Then

$$\begin{aligned} \frac{\partial}{\partial z}(z(r,\varphi), \vartheta(r,\varphi)) = \frac{\partial}{\partial z}(r,\varphi) = \left.\frac{d}{dt}\right|_{t=0} (\cos\varphi, \sin\varphi, r+t) = \\ = \left.\frac{d}{dt}\right|_{t=0} f(e^{(r+t)/\sqrt{2}}(\cos\varphi, \sin\varphi, 1)) = Df(\frac{\partial}{\partial r}(r,\varphi)), \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \vartheta}(z(r,\varphi), \vartheta(r,\varphi)) = \frac{\partial}{\partial \vartheta}(r,\varphi) = \left.\frac{d}{dt}\right|_{t=0} (\cos(\varphi+t), \sin(\varphi+t), r) = \\ = \left.\frac{d}{dt}\right|_{t=0} f(e^{(r)/\sqrt{2}}(\cos(\varphi+t), \sin(\varphi+t), 1)) = Df(\frac{\partial}{\partial \varphi}(r,\varphi)). \end{aligned}$$

Now observe that both metrics on C and M are given by the identity matrices, so f is indeed an isometry.

P4.4. Let G be a Lie group with bi-invariant metric. Show that it has non-negative sectional curvature.

Solution:

Let $g \in G$ and take any $x, y \in T_g G$. Take left-invariant vector fields X, Y such that $X(g) = x$, $Y(g) = y$. We want to compute $\langle R(X,Y)Y, X\rangle$ and show that it is never less than zero.

Recall from HW 3.1, that for left-invariant vector fields $\nabla_X Y = \frac{1}{2}[X,Y]$. Thus, we can compute $R(X,Y)Y$:

$$R(X,Y)Y = \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y = \frac{1}{2}\nabla_X \underbrace{[Y,Y]}_{=0} - \frac{1}{2}\nabla_Y [X,Y] - \nabla_{[X,Y]} Y =$$

$$-\frac{1}{4}[Y,[X,Y]] - \frac{1}{2}[[X,Y],Y] = -\frac{1}{4}[[X,Y],Y]$$

Using the equality $\langle [U,X], V\rangle = -\langle U, [V,X]\rangle$ proved in Corollary 6.18, we obtain

$$\langle R(X,Y)Y, X\rangle = \langle -\frac{1}{4}[[X,Y],Y], X\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle = \frac{1}{4}\|[X,Y]\|^2 \geq 0.$$