Durham University
Pavel Tumarkin Epiphany 2020

Riemannian Geometry IV, Homework 1 (Week 11)

Due date for starred problems: **Friday, January 31**.

1.1. ($\star$) Consider the upper half-plane $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the metric

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{y} \end{pmatrix}$$

(a) Show that all the Christoffel symbols are zero except $\Gamma^2_{22} = -\frac{1}{2y}$.

(b) Show that the vertical segment $x = 0$, $\varepsilon \le y \le 1$ with $0 < \varepsilon < 1$ is a geodesic curve when parametrized proportionally to arc length.

(c) Show that the length of the segment $x = 0$, $\varepsilon \le y \le 1$ with $0 < \varepsilon < 1$ tends to 2 as ε tends to zero.

(d) Show that (M, g) is not geodesically complete.

1.2. Let G, H be Lie groups. A map $\varphi : G \to H$ is called a *homomorphism (of Lie groups)* if it is smooth and it is a homomorphism of abstract groups.

Denote by $\mathfrak{g}, \mathfrak{h}$ Lie algebras of G and H, and let $\varphi : G \to H$ be a homomorphism.

(a) Show that the differential $D\varphi(e) : T_eG \to T_eH$ induces a linear map $D\varphi : \mathfrak{g} \to \mathfrak{h}$, where $D\varphi(X)$ for $X \in \mathfrak{g}$ is the unique left-invariant vector field on H such that $D\varphi(X)(e) = D\varphi(X(e))$.

(b) Show that for any $g \in G$

$$L_{\varphi(g)} \circ \varphi = \varphi \circ L_g$$

(c) Show that for any $X \in \mathfrak{g}$ and $g \in G$

$$D\varphi(X)(\varphi(g)) = D\varphi(X(g))$$

(d) Show that $D\varphi : \mathfrak{g} \to \mathfrak{h}$ is a *homomorphism of Lie algebras*, i.e. a linear map satisfying $D\varphi([X, Y]) = [D\varphi(X), D\varphi(Y)]$ for any $X, Y \in \mathfrak{g}$.

1.3. Let $S^2 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$.

Show that there exists no group operation on S^2 such that S^2 with this group operation and some smooth structure becomes a Lie group.