Durham University
Pavel Tumarkin

Epiphany 2020

Riemannian Geometry IV, Solutions 2 (Week 12)

2.1. ($\star$) Let $H_3(\mathbb{R})$ be the set of 3×3 unit upper-triangular matrices (i.e. the matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_1, x_2, x_3 \in \mathbb{R}$).

(a) Show that $H_3(\mathbb{R})$ is a group with respect to matrix multiplication. This group is called the *Heisenberg group*.

(b) Show that the Heisenberg group is a Lie group. What is its dimension?

(c) Prove that the matrices

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis of the tangent space $T_e H_3(\mathbb{R})$ of the group $H_3(\mathbb{R})$ at the neutral element e.

(d) For each $k = 1, 2, 3$, find an explicit formula for the curve $c_k : \mathbb{R} \to H_3(\mathbb{R})$ given by $c_k(t) = \operatorname{Exp}(tX_k)$.

Solution:

(a) It is an easy computation to check the axioms of a group (i.e H_3 is closed under multiplication, there exists an obvious neutral element (3×3 identity matrix), there is an inverse element for each $h \in H_3$, associativity works as always in matrix groups).

(b) The matrix elements (x_1, x_2, x_3) give a global chart on H_3, so H_3 is a smooth 3-manifold. The multiplication $g_1 g_2$ can be written as $(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1+y_1, x_2+y_2+x_1y_3, x_3+y_3)$, and the inverse element g_1^{-1} can be written as $(x_1, x_2, x_3)^{-1} = (-x_1, x_1x_3 - x_2, -x_3)$, which are smooth maps $H_3 \times H_3 \to H_3$ and $H_3 \to H_3$ respectively. Hence, H_3 is a Lie group.

(c) To see that the matrices X_i belong to $T_e H_3$ consider the paths $c_i(t) = I + X_i t \in H_3$. By definition, $\frac{\partial}{\partial x_i} = c_i'(t) = X_i$. So, $\{X_1, X_2, X_3\}$ is the basis of $T_e H_3$ since $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\}$ is a basis.

(d) Since $X_i^2 = 0$ for $i = 1, 2, 3$ we see that $\operatorname{Exp}(tX_i) = I + X_i t$.

2.2. (a) Let $A, B \in M_n(\mathbb{R})$, $[A, B] = 0$. Take $t \in \mathbb{R}$ and show that $\operatorname{Exp}(t(A + B)) = \operatorname{Exp}(tA)\operatorname{Exp}(tB)$ (in particular, you obtain that $\operatorname{Exp}(A + B) = \operatorname{Exp}(A)\operatorname{Exp}(B)$).

(b) Show that

$$\operatorname{Exp}\left(t \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Guess what would be the exponential of an $n \times n$-matrix of the same form (i.e., a Jordan block with zero eigenvalue).

(c) Show that

$$\operatorname{Exp}\left(t\begin{pmatrix} c & 1 & 0 & 0\\ 0 & c & 1 & 0\\ 0 & 0 & c & 1\\ 0 & 0 & 0 & c\end{pmatrix}\right) = e^{tc}\begin{pmatrix} 1 & t & t^2/2 & t^3/6\\ 0 & 1 & t & t^2/2\\ 0 & 0 & 1 & t\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Solution:

(a) As in the previous exercise, expand both exponents $\operatorname{Exp}(tA)$ and $\operatorname{Exp}(tB)$ as power series and collect the coefficient of t^n in the product. The monomials involved will be of type $\frac{(tA)^k(tB)^{n-k}}{k!(n-k)!}$, so the monomial containing t^n in the product will be

$$\sum_{k=0}^{n}\frac{(tA)^k(tB)^{n-k}}{k!(n-k)!} = \sum_{k=0}^{n} t^n\frac{A^kB^{n-k}}{k!(n-k)!} = \frac{t^n}{n!}\sum_{k=0}^{n}A^kB^{n-k}\frac{n!}{k!(n-k)!} = \frac{t^n}{n!}(A+B)^n$$

(b) Let $A = \begin{pmatrix} 0 & t & 0 & 0\\ 0 & 0 & t & 0\\ 0 & 0 & 0 & t\\ 0 & 0 & 0 & 0\end{pmatrix}$. We have

$$A^2 = \begin{pmatrix} 0 & 0 & t^2 & 0\\ 0 & 0 & 0 & t^2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 0 & 0 & t^3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad A^k = 0 \qquad \text{for all } k \geq 4.$$

So the power series $\operatorname{Exp}(A)$ terminates after 4 terms and we conclude that

$$\operatorname{Exp}(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 = \begin{pmatrix} 1 & t & t^2/2 & t^3/(3!)\\ 0 & 1 & t & t^2/2\\ 0 & 0 & 1 & t\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

(c) Let $B = tcI$, where I denotes the 4×4 identity matrix, and let A be as in (a). Then we have $\operatorname{Exp}(B) = e^{tc}I$ and A and B commute. This implies that

$$\operatorname{Exp}\left(t\begin{pmatrix} c & 1 & 0 & 0\\ 0 & c & 1 & 0\\ 0 & 0 & c & 1\\ 0 & 0 & 0 & c\end{pmatrix}\right) = \operatorname{Exp}(A+B) = \operatorname{Exp}(B)\operatorname{Exp}(A) = e^{tc}\begin{pmatrix} 1 & t & t^2/2 & t^3/(3!)\\ 0 & 1 & t & t^2/2\\ 0 & 0 & 1 & t\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

2.3. The *special unitary group* $SU_n \subset M_n(\mathbb{C})$ consists of $n \times n$ matrices A with complex entries and unit determinant satisfying the equation $\bar{A}^tA = I = A\bar{A}^t$.

(a) Show that SU_n forms a group under matrix multiplication.

(b) Show that SU_2 consists of all matrices of the form

$$\begin{pmatrix} z & w\\ -\bar{w} & \bar{z}\end{pmatrix}, \quad z, w \in \mathbb{C}, \quad |z|^2 + |w|^2 = 1.$$

(c) Show that SU_2 is a smooth (real) manifold. Find its dimension.

(d) Show that SU_2 is a Lie group.

(e) Find the Lie algebra $\mathfrak{su}_2$ of SU_2 as a subspace of $M_2(\mathbb{C})$. Find any basis $\{v_1, v_2, v_3\}$ of $\mathfrak{su}_2$. Compute explicitly the left-invariant vector fields X_1, X_2, X_3 on SU_2 such that $X_i(I) = v_i$.

Solution:

(a) Let $A, B \in SU_n$. Then

$$(\overline{AB})^t(AB) = \bar{B}^t\bar{A}^tAB = \bar{B}^t(\bar{A}^tA)B = \bar{B}^tB = I,$$

so $AB \in SU_n$. Also, $\det\bar{A}^t \det A = \det I = 1$ and $\det\bar{A}^t = \overline{\det A}$, which implies $|\det A| = 1 \neq 0$. Thus, A^{-1} exists. Now observe that $(\bar{A}^t)^{-1}A^{-1} = (A\bar{A}^t)^{-1} = I$, so $A^{-1} \in SU_n$.

(b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{C}$. Then, computing $\bar{A}^t A$, we see that $A \in SU_2$ if and only if the following equations hold:

$$|a|^2 + |b|^2 = 1, \quad |c|^2 + |d|^2 = 1, \quad a\bar{c} + b\bar{d} = 0, \quad ad - bc = 1.$$

Multiplying the last two equations by c and $\bar{d}$ respectively and adding them to each other, we see that $a(|c|^2 + |d|^2) = \bar{d}$, which implies $a = \bar{d}$. This, in its turn, immediately implies that $c = -\bar{b}$.

Thus, we proved that every $A \in SU_2$ has required form. Conversely, it is clear that every matrix of such form has unit determinant and satisfies $\bar{A}^t A = I$.

(c) We can embed SU_2 in $\mathbb{R}^4$ with coordinates $(x_1, \ldots, x_4)$ by writing $z = x_1 + ix_2$ and $w = x_3 + ix_4$. Thus, $SU_2 = f^{-1}(0)$ for $f : \mathbb{R}^4 \to \mathbb{R}$, $f(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1$. Since 0 is a regular value, SU_2 is a 3-dim smooth manifold (actually, the description above shows that SU_2 is the 3-dim sphere S^3).

(d) The multiplication and inverse are polynomials in the entries so they are clearly smooth.

(e) Let $A(t) = \begin{pmatrix} x_1(s) + ix_2(s) & x_3(s) + ix_4(s) \\ -x_3(s) + ix_4(s) & x_1(s) - ix_2(s) \end{pmatrix}$ be a curve in SU_2, $A(0) = I$. Differentiating the equation $x_1^2(s) + x_2^2(s) + x_3^2(s) + x_4^2(s) = 1$ at $s = 0$, we obtain $x_1'(0) = 0$. In other words,

$$\mathfrak{su}_2 = T_I SU_2 = \left\{ \begin{pmatrix} xi & w \\ -\bar{w} & -xi \end{pmatrix} \mid x \in \mathbb{R},\, w \in \mathbb{C},\ x^2 + |w|^2 = 1 \right\}.$$

We can take as a basis of $\mathfrak{su}_2$, for example, matrices

$$v_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

(this particular choice of signs can be explained by the fact that the matrices $\sigma_1 = iv_1, \sigma_2 = iv_2, \sigma_3 = iv_3$ are Pauli matrices you could meet in Quantum Mechanics).

To construct left-invariant fields X_i recall from Example 6.3 that for matrix groups $X_i(g) = gX_i(I)$. Thus, for $g = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$, we have

$$X_1(g) = \begin{pmatrix} -iw & -iz \\ -i\bar{z} & i\bar{w} \end{pmatrix}, \quad X_2(g) = \begin{pmatrix} w & z \\ \bar{z} & \bar{w} \end{pmatrix}, \quad X_3(g) = \begin{pmatrix} -iz & iw \\ i\bar{w} & i\bar{z} \end{pmatrix}$$